REMOVABLE EDGES IN 4-CONNECTED GRAPHS

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Removable Edges in 4-Connected Graphs

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Prof. dr. ir. H. J. Broersma (Promotor)

Prof. dr. Xueliang Li (Co-promotor)

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Preface

It took almost four years to complete this dissertation which is about the theory on removable edges in 4-connected graphs. After an introductory chapter the readers will find six chapters that contain these topics within this research field. These topics have more or less strong connections with each other. Some of results have been published in journals, see the following list.

Papers underlying this thesis

[1] J. Wu, X. Li, Removable Edges in Longest Cycles of 4-Connected Graphs, *Graphs & Combin.* (2004)20:413-422 (Chapter 3)

[2] J. Wu, X. Li and L. Wang, Removable Edges in a Cycle of a 4-Connected Graph, *Discrete Mathematics* 287(2004), 103-111.(Chapter 4)

[3] J. Wu, X. Li and J. Su, The Number of Removable Edges in 4-connected Graphs, *Journal of Combin.Theory Ser.B,* **92** (2004), 13-40. (Chapter 6)

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Jichang Wu
September 2009, Enschede

Chapter 1

Introduction

Research on structural characterizations of graphs is a very popular topic in graph theory. The concepts of contractible edges and removable edges of graphs are powerful tools to study the structure of graphs and to prove properties of graphs by induction.

In 1961, Tutte [40] gave a structural characterization of 3-connected graphs by using the existence of contractible edges and removable edges. He proved that every 3-connected graph with order at least 5 contains contractible edges, and any a simple 3-connected graph nonisomorphic to K_4 can be obtained from the wheels by sequentially adding edges and by what Tutte called splitting vertices, which is Tutte's famous Wheel Theorem. This is the earliest result concerning the concept of contractible edges and removable edges. In addition to Tutte's results on the construction of 3-connected graphs, Barnette [4, 5, 6] gave three different methods to construct 3-connected graphs by using removable edges, 3-cycle contraction and cycle-contraction. As a supplement of Tutte's result, in 1982, Negami [28] obtained the following results: Let K be a 3-connected graph which is not a wheel. Then G is a 3-connected graph which can be contracted to K if and only if G can be obtained from K by repeatedly adding and splitting edges . In 1978, Mader [22] gave a reduction method to construct k-edge-connected graphs. In 1979, Chaty and Chein [8] gave a method to constructe minimally 2-edge-connected graphs. In 1989, Zhu [46] gave a method on how to construct a minimally k-edge-connected graph. Zhang, Guo and Chen [47] described the construction of critically k-edge-connected graphs. In 1994, based on the work of Habib and Peroche [14], Peroche etc. [31] succeeded to construct the minimally 4-edge-connected graphs. In 2003, Hennayake etc. [15] gave a method to construct a minimally (k, k)-edge-connected graphs, where a connected graph G is $(k; k)$-edge-connected if the k-edge-connectivity of G is at least k. Recently, Kriesell [18] presented a method to construct the class C of finite simple 3-connected triangle-free graphs from the 3-regular complete bipartite graph $K_{3,3}$ and the skeleton of a 3-dimensional cube.

A well-known application of the existence of contractible edges in 3-connected graphs was given by Thomassen [39]. By induction he gave a very simple proof for the three well-known theorems on planar graphs, i.e., Kuratowski's Theorem: a graph is planar if and only if it does not contain any subgraph homeomorphic to K_5 or $K_{3,3}$; Fary's Theorem: every planar graph has a plane linear representation; and Tutte's Theorem: every 3-connected graph has a plane convex representation. The earlier proofs of the three theorems were very complicated and tedious.

Another successful application of contractible edges is as follows. In 1974, Lovász [21] posed the conjecture: let G be an n-connected graph and F be a set of independent edges of G such that $|F| = n$. If n is even or $G - F$ is connected, then G has a cycle containing all the edges of F. Ando, Enomoto and Saito [2] showed that the conjecture is true for $n = 3$ by using contractible edges in 3-connected graphs.

From the above examples we can see the importance of studying the existence and distribution of contractible edges and removable edges of graphs. Holton, Jackson, Saito and Wormald [16] studied the number of removable edges in a 3-connected graph and their distribution. Su [32] obtained a sharp lower bound on the number of removable edges in 3-connected graphs and also gave a structural characterization of 3-connected graphs for which the lower bound is sharp. Fouquet, Thuiller [12] studied removable edges in 3-regular graph. Contractible edges and removable edges graphs have been studied extensively in the literatures, see also [23-30, 32, 33], especially [19] for a survey on contractible edges.

In 1974, Slater [36] presented a method for constructing 4-connected graphs. He proved that a 4-connected graph can be obtained from K_5 by using the following operations repeatedly: (1) adding edges; (2) 4-soldering; (3) 4-point-splitting; (4) 4-line-splitting; (5) 3-fold-4-point-splitting. Later, Yin [43] gave a more convenient method to construct 4-connected graphs by using removable edges and contractible edges. Yin proved that there always exist removable edges in a 4-connected graph G, unless G is a 2-cyclic graph with order 5 or 6. A 2-cyclic graph G of order n is defined to be the square of the cycle C_n, C_n^2 is obtained from C_n by adding edges between all pairs of vertices of C_n which are at distance 2 in C_n. See Figure 1.1.

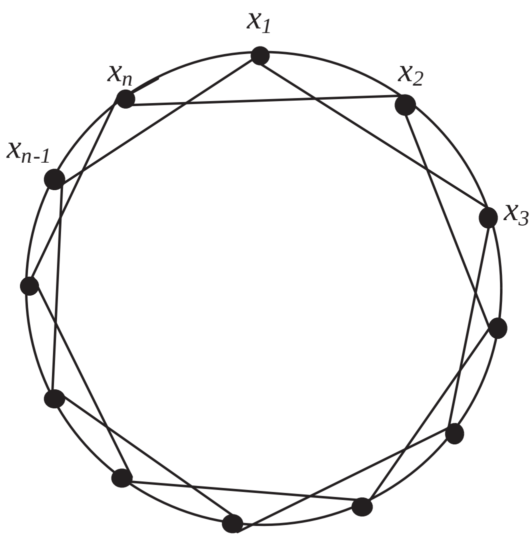

Figure 1.1:

He also showed that a 4-connected graph can be obtained from a 2-cyclic graph by the following four operations: (i) adding edges, (ii) splitting vertices, (iii) adding vertices and removing edges, and (iv) extending vertices. Recently Ando, Egawa, Kawarabayashi and Kriesell [3] studied the number of contractible edges in 4-connected graph. In this thesis we shall focus on the study of removable edges in 4-connected graphs.

In Chapter 2 we introduce some results obtained by Yin. Since those results are published in Chinese, for convenience, we repeat them in Chapter 2 together with their proofs. However, we use some new ideas in some of those proofs.

In Chapter 3 we study how many removable edges may exist in a cycle of a 4-connected graph, and we give examples to show that our results are in some sense the best possible.

In Chapter 4 we obtain results on removable edges in a longest cycle of a 4-connected graph. We also show that for a 4-connected graph G of minimum degree at least 5 or girth at least 4, any edge of G is removable or contractible.

In Chapter 5 we study the distribution of removable edges on a Hamilton cycle of a 4-connected graph, and show that our results cannot be improved in some sense.

In Chapter 6 we prove that every 4-connected graph of order at least six except C_6^2 has at least $(4|G|+16)/7$ removable edges. We also give a structural characterization of 4-connected graphs for which the lower bound is sharp.

In Chapter 7 we study how many removable edges there are in a spanning tree of a 4-connected graph and how many removable edges exist outside a cycle of a 4-connected graph. We also give examples to show that our results can not be improved in some sense.

1.1 Some Basic Notations and Definitions

In this section we give some basic terminologies, notations and definitions which appear in this dissertation.

Without specific statement, in this thesis G always denotes a 4-connected graph. The vertex set and edge set of G are denoted, respectively, by $V(G)$ and $E(G)$. The order and size of G are denoted, respectively, by $|G|$ and $|E(G)|$. For $x \in V(G)$, we simply write $x \in G$. The neighborhood of $x \in G$ is the set of all vertices of G that are adjacent to x, denoted by $\Gamma_G(x)$. The degree of x is$|\Gamma_G(x)|$, and is denoted by $d_G(x)$. If x and y are the two end-vertices of an edge e, we write $e = xy$. For a nonempty subset N of $V(G)$, the induced subgraph by N in G is denoted by $[N]$. Let $A, B \subset V(G)$ such that $A \neq \emptyset \neq B$ and $A \cap B = \emptyset$. We define $[A, B] = \{xy \in E(G) \mid x \in A, y \in B\}$. If H is a subgraph of G, we say that G contains H. For a subset S of $V(G)$, $G - S$ denotes the graph obtained by deleting all the vertices in S from G together with all the incident edges. If $G - S$ is disconnected, we say that S is a vertex-cut of G. If $|S| = s$ for such an S, we say that S is an s-vertex-cut. A cycle of G with l vertices is simply called an l-cycle of G. The girth of a graph G is the smallest length of among cycles of G, and denoted by $g(G)$.

Definition 1.1.1. Let G be a 4-connected graph. For an edge e of G, we perform the following operations on G: First, delete the edge e from G, resulting in the graph $G - e$; Second, for each vertex x of degree 3 in $G - e$, delete x from $G - e$ and then completely connect the 3 neighbors of x by a triangle. (See Figure 1.2). If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by $G \ominus e$. If $G \ominus e$ is still 4-connected, then the edge e is called *removable*; otherwise, e is called *unremovable*. The set of all removable edges of G is denoted by $E_R(G)$, whereas the set of unremovable edges of G is denoted by $E_N(G)$. The numbers of removable edges and unremovable edges are denoted by $e_R(G)$ and $e_N(G)$, respectively.

Definition 1.1.2. A *2-cyclic graph* G of order n is defined to be the square of the cycle C_n, i.e., G can be obtained from C_n by adding edges between all

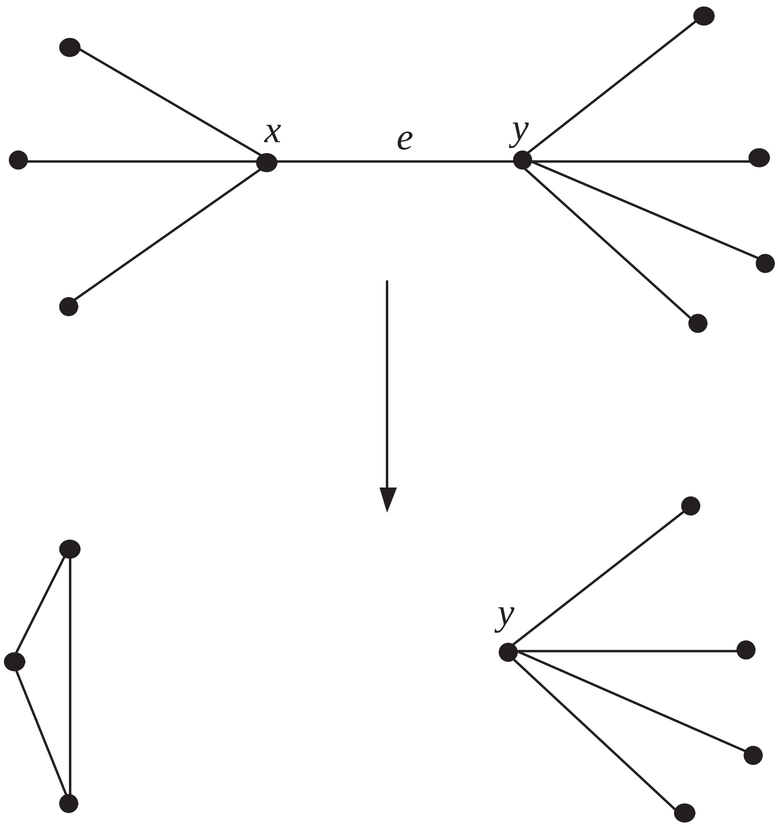

Figure 1.2:

pairs of vertices of C_n which are at distance 2 in C_n.

Definition 1.1.3. Let G be a 4-connected graph, and suppose that for $e = xy \in E(G)$ and $S \subset V(G)$ such that $|S| = 3$, $G - e - S$ has exactly two (connected) components, say A and B, such that $|A| \geq 2$ and $|B| \geq 2$. Then we say that (e, S) is a *separating pair* and $(e, S; A, B)$ is a *separating group*, in which A and B are called the *edge-vertex-cut fragments*. See Figure 1.3.

Definition 1.1.4. Let G be a 4-connected graph, for $e = xy \in E(G)$ and $S \subset V(G)$ such that $|S| = 3$, $G - e - S$ has exactly two (connected) components, say A and B with $|A| \geq 2$ and $|B| \geq 2$. If $|A| = 2$, then A is called an *edge-vertex-cut atom*. For an edge-vertex-cut atom A, let $A = \{x, z\}$ and $S = \{a, b, c\}$. If $ax, bx \in E(G), cx \notin E(G)$, then A is called a *1-edge-vertex-cut*

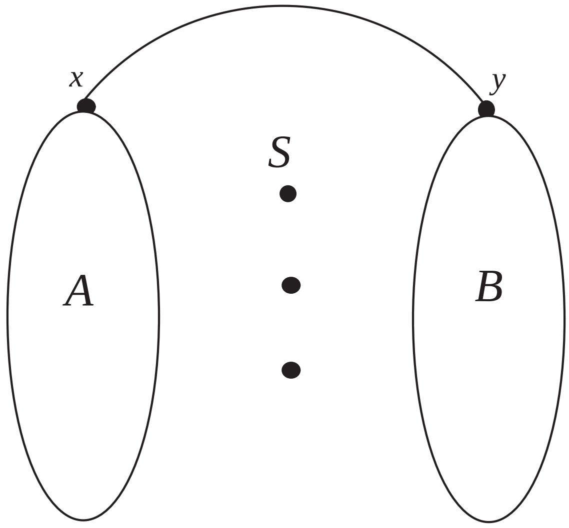

Figure 1.3:

atom; whereas if $ax, bx, cx \in E(G)$, then A is called a *2-edge-vertex-cut atom*. Both a 1-edge-vertex-cut atom and a 2-edge-vertex-cut atom are called *2-atom*.

Since in a 4-connected graph every vertex has degree at least 4, it is easy to see that if A is an edge-vertex-cut atom, then A is either a 1-edge-vertex-cut atom or a 2-edge-vertex-cut atom.

Definition 1.1.5. Let G be a 4-connected graph, $E_0 \subseteq E_N(G)$ such that $E_0 \neq \emptyset$ and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$ and $y \in B$. If $xy \in E_0$, then A and B are called *E_0-edge-vertex-cut fragments*. An E_0-edge-vertex-cut fragment is called an *E_0-edge-vertex-cut end-fragment* of G if it does not contain any other E_0-edge-vertex-cut fragment of G as a proper subset. Similarly, if $|A| = 2$, then A is called an *E_0-edge-vertex-cut atom*.

It is easy to see that any E_0-edge-vertex-cut fragment of G contains such an end-fragment.

Definition 1.1.6. Let xy be an edge of a 4-connected graph G, and let G' be the simple graph obtained from G by first removing the edge xy, then identifying x and y by introducing a new vertex v_{xy} and finally making the new vertex v_{xy} adjacent to all vertices that are originally adjacent to x or y. We call the edge xy *contractible* if G' is still 4-connected; otherwise, it is called *non-contractible*. The set of all contractible edges of G is denoted by $E_C(G)$.

Let G be a 4-connected noncomplete graph. Then it is easy to see that an edge $e = xy$ is non-contractible if and only if there exists a vertex-cut of G with 4 vertices containing x and y.

1.2 Terminology and Notations for Subgraphs with Special Structures

For convenience, we introduce the following special terminology and notations for subgraphs with special structures in a graph G.

Definition 1.2.1. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2,$ $x_2x_3,\ x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$. If H satisfies the following conditions

(i) $d_G(a) = d_G(x_i) = 4$ for $i = 1, 2, 3, 4$.
(ii) $ax_1, ax_2, ax_3, ax_4 \in E_N(G)$ and $x_1x_2, x_2x_3, x_3x_4, x_4x_1 \in E_R(G)$.

then H is called a *helm*, and the edges ax_i, for $i = 1, 2, 3, 4$, are called *inner edges* of H, the vertices a, x_i, for $i = 1, 2, 3, 4$, of a helm H are called *inner vertices* of H. See Figure 1.4.

Definition 1.2.2. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{a, b, x_1, x_2, \cdots, x_{l+3}\}$ and $E(H) = \{x_1x_2, x_2x_3, \cdots, x_{l+2}x_{l+3}, ax_2,$ $ax_3, \cdots,\ ax_{l+2},\ bx_2, bx_3, \cdots, bx_{l+2}\}$ with $l \geq 1$. If H satisfies the following conditions

(i) $x_ix_{i+1} \in E_N(G)$, for $i = 1, 2, \cdots, l+2$,
(ii) $ax_j, bx_j \in E_R(G)$, for $j = 2, 3, \cdots, l+2$,

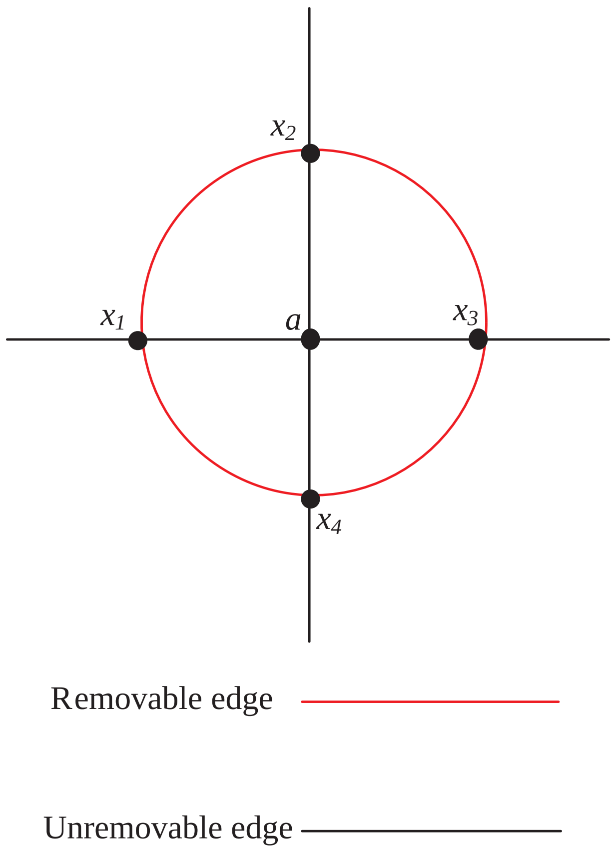

Figure 1.4:

(iii) $d_G(x_j) = 4$, for $j = 2, 3, \cdots, l+2$.

then H is called an *l-bi-fan*.

An l-bi-fan H is said to be maximal if $\Gamma_G(x_1) \neq \{a, b, x_2, u\}$ and $\Gamma_G(x_{l+3}) \neq \{a, b, x_{l+2}, v\}$ for any $u, v \in G$. The edges $x_j x_{j+1}$ for $j = 2, 3, \cdots, l+1$ of an l-bi-fan H are called *inner edges* of H, and the vertices x_j for $j = 2, 3, \cdots, l+1$ of an l-bi-fan H are called *inner vertices* of H. See Figure 1.5.

Definition 1.2.3. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, \cdots, x_{l+2}, y_1, y_2, \cdots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$, where $E_1(H) = \{x_1x_2, x_2x_3, \cdots, x_{l+1}x_{l+2}, y_1y_2, y_2y_3, \cdots, y_{l+1}y_{l+2}\}$ and

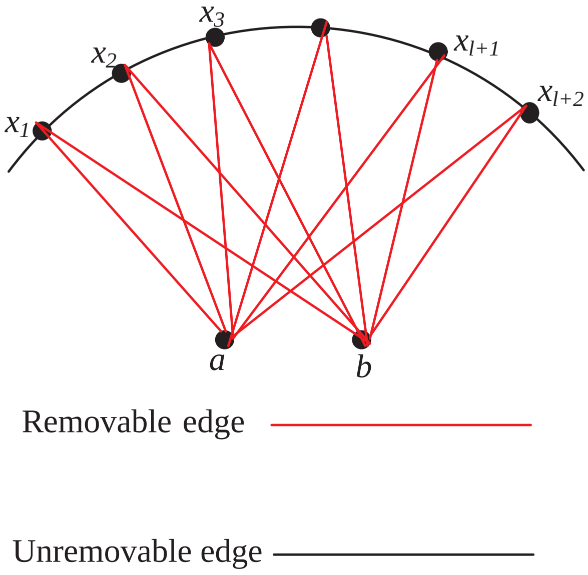

Figure 1.5:

$E_2(H) = \{y_1x_2,\ x_2y_2,\ y_2x_3, \cdots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}\}$. Then, H is called an *l-belt* if the following conditions are satisfied

(i) $E_1(H) \subseteq E_N(G)$ and $E_2(H) \subseteq E_R(G)$,
(ii) $d(x_i) = d(y_j) = 4$, for $i = 2, 3, \cdots, l+1; j = 2, 3, \cdots l+1$.

An l-belt H is said to be maximal if $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$ and $\Gamma_G(x_{l+2}) \neq \{x_{l+1}, y_{l+1}, y_{l+2}, v\}$ for any $u, v \in G$. The edges x_ix_{i+1}, y_jy_{j+1} for $i = 2, 3, \cdots, l+1; j = 1, 2, \cdots, l$ of an l-belt or a maximal l-belt H are called *inner edges* of H, and the vertices x_i, y_j for $i = 2, 3, \cdots, l+1; j = 2, 3, \cdots l+1$ of an l-belt H are called *inner vertices* of H. See Figure 1.6.

Definition 1.2.4. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, \cdots, x_{l+2}, x_{l+3}, y_1, y_2, \cdots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$ where $E_1(H) = \{x_1x_2, x_2x_3, \cdots, x_{l+2}x_{l+3}, y_1y_2, y_2y_3, \cdots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \cdots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}, x_{l+2}y_{l+2}\}$ with $l \geq 1$ with $l \geq 1$. Then, H is called an *l-co-belt* if the following conditions are satis-

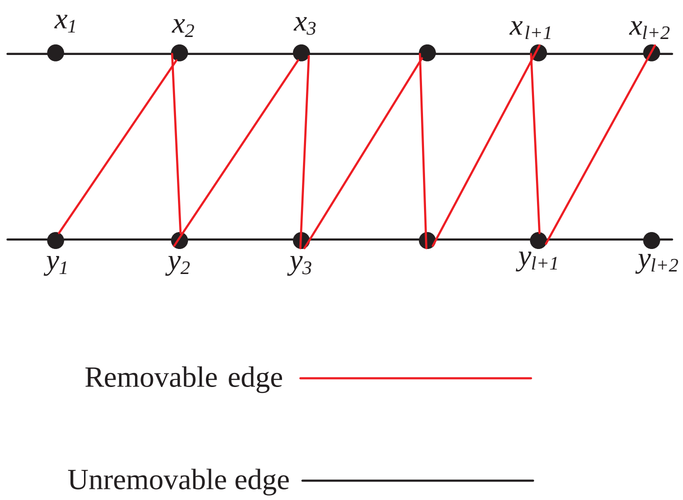

Figure 1.6:

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(i) $E_1(H) \subseteq E_N(G)$ and $E_2(H) \subseteq E_R(G)$,
(ii) $d_G(x_i) = d_G(y_j) = 4$ for $i = 2, 3, \cdots, l+2; j = 2, 3, \cdots l+1$.

An l-co-belt H is said to be maximal if $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$ and $\Gamma_G(y_{l+2}) \neq \{x_{l+2}, y_{l+1}, x_{l+3}, v\}$, for any $u, v \in G$. The edges x_ix_{i+1}, y_jy_{j+1}, for $i = 2, 3, \cdots, l+1; j = 1, 2, \cdots, l+1$, of an l-co-belt H are called *inner edges* of H, and the vertices x_i, y_j for $i = 2, 3, \cdots, l+2; j = 2, 3, \cdots l+1$ of an l-co-belt H are called *inner vertices* of H. See Figure 1.6.

Definition 1.2.5. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(H) = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. Then H is called a *W-framework* if the following conditions are satisfied:

(i) $x_ix_{i+1} \in E_N(G)$, for $i = 1, 2$,
(ii) $d_G(x_2) = d_G(y_2) = d_G(y_3) = 4$,
(iii) $y_2y_3, x_1y_2, x_2y_2, x_2y_3, x_3y_3 \in E_R(G)$.

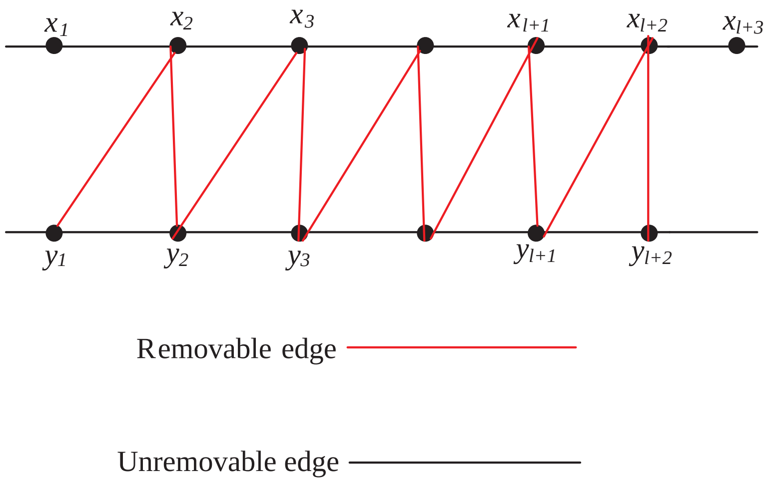

Figure 1.7:

The edges x_1x_2, x_2x_3 of a W-framework H are called *inner edges* of H, the vertex x_2 of a W-framework H is called the *inner vertex* of H. See Figure 1.8.

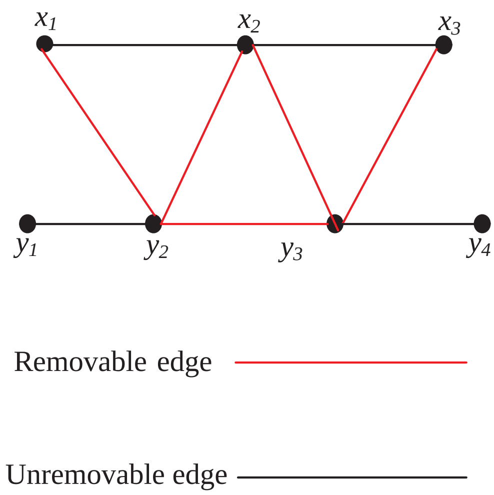

Figure 1.8:

Definition 1.2.6. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(H) = \{x_1x_2, x_2x_3, x_1x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. Then H is called a *W'-framework* if the following conditions are satisfied:

(i) $x_i x_{i+1} \in E_N(G)$, for $i = 1, 2$,
(ii) $d_G(x_2) = d_G(x_3) = d_G(y_2) = d_G(y_3) = 4$ and $d_G(x_1) \geq 5$,
(iii) $y_2y_3, x_1y_2, x_2y_3, x_3y_3, x_1x_3 \in E_R(G), x_2y_2 \in E_N(G)$.

The edges x_1x_2, x_2x_3, x_2y_2 of a W'-framework H are called *inner edges* of H, the vertices x_2, x_3 of a W'-framework H are called *inner vertices* of H. See Figure 1.9.

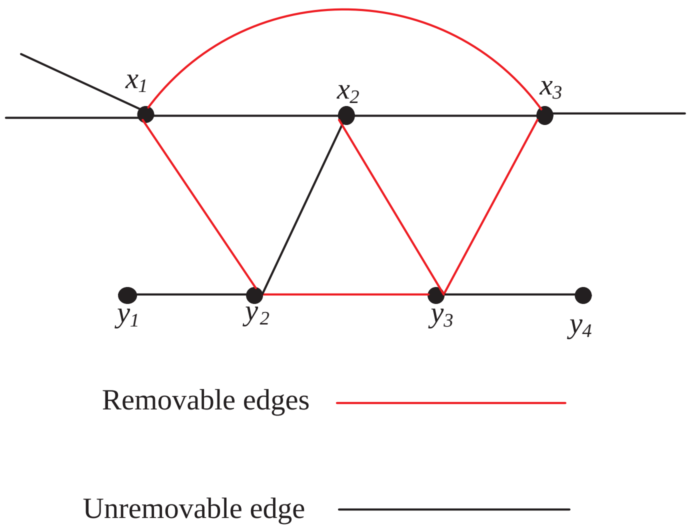

Figure 1.9:

1.3 Results on Removable Edges in 4-Connected Graphs

First of all, we list some known results on removable edges of 4-connected graphs, which can be found in [43].

Theorem 1.3.1. (Yin 1999) *Let G be a 4-connected graph with $|G| \geq 7$. An edge e of G is unremovable if and only if there is a separating pair (e, S), or a separating group $(e, S; A, B)$ in G.*

Theorem 1.3.2. (Yin 1999) *Let G be a 4-connected graph with $|G| \geq 8$ and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$ and $|A| \geq 3$. Then every edge in $[\{x\}, S]$ is removable.*

Corollary 1.3.1. (Yin 1999) *Let G be a 4-connected graph with $|G| \geq 8$. Then every triangle of G contains at least one removable edge.*

Theorem 1.3.3. (Yin 1999) *Let G be a 4-connected graph with $|G| \geq 7$. If for an unremovable edge xy, i.e., $xy \in E_N(G)$, there is a separating group $(xy, S; A, B)$, then all the edges in $E([S])$ are removable, i.e., $E([S]) \subset E_R(G)$.*

In addition, Yin studied the number of removable edges and contractible edges. Let H be the set of contractible edges or removable edges of G, and let k denote the number of *helms*, which are contained in the 4-connected graphs. Then the following result holds:

Theorem 1.3.4. (Yin 1999) *Let G be a 4-connected graph with order n, $(n \geq 5)$, $G \neq C_5^2$ and $G \neq C_6^2$, m is the number of vertices of degree four, then $|H| \geq \lceil (3n + 7k - 2m)/2 \rceil \geq \lceil n/2 \rceil$.*

In this dissertation we first study the distribution of removable edges in some special subgraphs of a 4-connected graph.

In Chapter 3 we study the distribution of removable edges in a cycle in a 4-connected graph. For this purpose we need the following technical lemma the proof of which appears in Chapter 3.

Lemma 1.3.1. *Suppose that G is a 4-connected graph, $(xy, S; A, B)$ is a separating group of G such that $x \in A, y \in B, S = \{a, b, c\}$ and A is a 1-edge-vertex atom, say $A = \{x, z\}$. Then precisely one of the following conclusions holds:*

(i) $ax, bx, zx \in E_R(G)$.

(ii) $ax \in E_N(G), d(x) = d(z) = 4, bx, zx, az \in E_R(G), zc \in E_N(G)$.

(iii) $ax \in E_N(G), ay \in E_R(G)$. *Moreover, if* $d(a) = 4, d(y) \geq 5$, *then* $az, zb, zx, by \in E_R(G), bx \in E_N(G)$; *if* $d(a) \geq 5, d(y) = 4$, *then* $by, bx, bz, az \in E_R(G), zx \in E_N(G)$; *if* $d(a) = d(y) = 4$, *then* $az, bz, by \in E_R(G), bx, zx \in E_N(G)$; *if* $d(a) \geq 5, d(y) \geq 5$, *then* $az, zx, bx, by \in E_R(G)$.

(iv) $ax, bx, ac, bc \in E_R(G), zx, zc \in E_N(G), \{za, zb\} \cap E_N(G) \neq \emptyset, d(x) = d(c) = d(z) = 4$. *If* $za \in E_N(G)$, *then the following conclusion holds:* $d(b) = 4$, *and if* $d(a) = 4$, *then* $bz \in E_N(G)$; *if* $d(a) \geq 5$, *then* $bz \in E_R(G)$. *If* $bz \in E_N(G)$, *then the following conclusion holds:* $d(a) = 4$, *and if* $d(b) = 4$, *then* $az \in E_N(G)$; *if* $d(b) \geq 5$, *then* $az \in E_R(G)$.

(v) $ax, bx, az, bz \in E_R(G), xz \in E_N(G), d(x) = d(z) = 4$.

(vi) $bx \in E_N(G), by \in E_R(G)$. *Moreover, if* $d(b) = 4, d(y) \geq 5$, *then* $bz, za, zx, ay \in E_R(G), ax \in E_N(G)$; *if* $d(b) \geq 5, d(y) = 4$, *then* $ay, ax, az, bz \in E_R(G), zx \in E_N(G)$; *if* $d(b) = d(y) = 4$, *then* $bz, az, ay \in E_R(G), ax, zx \in E_N(G)$; *if* $d(b) \geq 5, d(y) \geq 5$, *then* $bz, zx, ax, ay \in E_R(G)$.

(vii) $bx \in E_N(G), d(x) = d(z) = 4, ax, zx, bz \in E_R(G), zc \in E_N(G)$.

From the above lemma, we directly obtain the following conclusion.

Corollary 1.3.2. *Let* G *be a 4-connected graph with* $(xy, S; A, B)$ *a separating group of* G *such that* $x \in A, y \in B, S = \{a, b, c\}$. *Let* A *be a 1-edge-vertex-cut atom, say* $A = \{x, z\}$, *If* $\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$, *then* x *is an inner vertex of one of the following subgraphs in* G: *helm,* l*-co-belt,* l*-belt,* W'*-framework,* W*-framework or* l*-bi-fan.*

For a 2-edge-vertex-cut atom, we get the following result of which proof is in Chapter 3.

Lemma 1.3.2. *Let* G *be a 4-connected graph,* $(xy, S; A, B)$ *a separating group of* G, A *a 2-edge-vertex-cut atom, say* $A = \{x, z\}$ *and* $S = \{a, b, c\}$. *Then* $ax, bx, cx, xz \in E_R(G)$.

For convenience we denote by $\Re$ the set of all helms, maximal l-bi-fans, maximal l-belts, maximal l-co-belts, W-frameworks and W'-frameworks of a graph G.

Definition 1.3.1. *Let C be a cycle of a 4-connected graph G and H a subgraph of G belonging to $\Re$. If C contains an inner vertex of H, then we say that C passes through H.*

In Chapter 3 we will prove that for a cycle in a 4-connected graph, the following conclusions hold.

Theorem 1.3.5. *Let G be a 4-connected graph and C a cycle of G. If C does not pass through any subgraph of G belonging to $\Re$, then there are at least two removable edges of G in C.*

Theorem 1.3.6. *Let G be a 4-connected graph and C a cycle of G. If C passes through only one subgraph of G belonging to $\Re$, then there exists at least one removable edge of G in C.*

We also present examples in Chapter 3 to show that in some sense the above two results are the best possible.

We obtain the following result on removable edges and contractible edges in Chapter 4:

Theorem 1.3.7. *Let G be a 4-connected graph with $|G| \geq 8$ such that $\delta(G) \geq 5$ or $g(G) \geq 4$. Then any edge of G is removable or contractible.*

For removable edges on a longest cycle in a 4-connected graph, we get the following results in Chapter 4.

Definition 1.3.2. *Let G be a 4-connected graph and H a subgraph of G. If $V(H) = \{u, v, x, z\}$, $E(H) = \{xz, ux, vx, uz, vz\}$ and $d(x) = d(z) = 4$, then H is called a bi-triangle, and x, z are called its inner vertices. If a cycle C of G contains the vertices u, v, x and z, we say that C passes through the bi-triangle*

H.. See Figure 1.10.

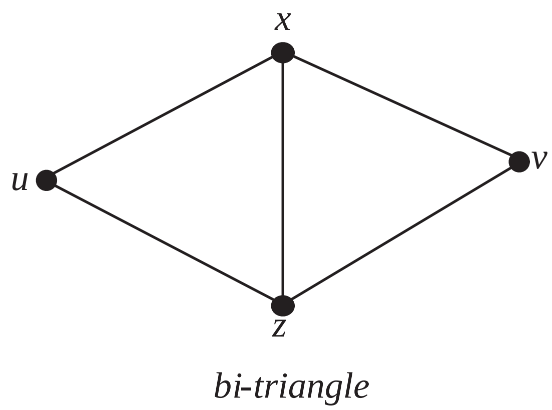

Figure 1.10:

Theorem 1.3.8. *Let G be a 4-connected graph with $|G| \geq 8$. If a longest cycle C of G does not pass through any bi-triangle, then C contains at least two removable edges.*

Theorem 1.3.9. *Let G be a 4-connected graph with $|G| \geq 8$. If a longest cycle C of G passes through at most one bi-triangle, then C contains at least one removable edge.*

In Chapter 5 we study the distributions of removable edges in Hamilton cycles in 4-connected Hamilton graphs. The following lemma of which proof can be found in Chapter 5 is necessary for our main results.

Lemma 1.3.3. *Let G be a 4-connected graph, $E_0 \subset E_N(G)$ and $E_0 \neq \emptyset$. Let $(xy, S; A, B)$ be a separating group of G such that $x \in A, y \in B, S = \{a, b, c\}, xy \in E_0$. If A is an E_0-edge-vertex end-fragment of G, and $|A| \geq 3$, then one of the following conclusions holds:*

(i) $(E(A) \cup [A, S]) \cap E_0 = \emptyset$.

(ii) *There exists a separating group $(x'y', S'; A', B')$ of G such that $x' \in A', y' \in B', x'y' \in E_0$, B' is a 1-edge-vertex-cut atom, and $|A \cap B'| = |B' \cap S| = 1$.*

(iii) *There exists a separating group $(xy', S'; A', B')$ of G such that $x \in A', y' \in B', xy' \in E_0$, $A \cap A' = \{x\}, |A \cap S'| = 1, A \cap B' = \{y'\}, |B' \cap S| = 2.$*

Based on the above lemma, we show the following result on removable edges in a Hamilton cycle of a 4-connected Hamilton graph in Chapter 5:

Theorem 1.3.10. *Let G be a 4-connected graph with $|G| \geq 7$, C a Hamilton cycle of G. If C does not pass through any 2-atom of G, then there are at least three removable edges on C.*

The following lemma of which proof is in Chapter 5 is used in the proof of the Theorem 1.3.11.

Lemma 1.3.4. *Let G be a 4-connected graph with $|G| \geq 7$, and let C be a cycle which exactly contains one inner vertex of some maximal l-bi-fan H, and C does not pass through any other subgraph belonging to $\Re$, then there are at least two removable edges on C.*

Theorem 1.3.11. *Let G be a 4-connected graph with $|G| \geq 7$, C a Hamilton cycle of G. If C passes through only one subgraph (excluding maximal l-belt or l-co-belt) belonging to $\Re$, and doesn't pass through any maximal l-belt or l-co-belt, there are at least two removable edges on C.*

In Chapter 6, we obtain a lower bound on the number of removable edges in a 4-connected graph, and give a structural characterization of 4-connected graphs for which the lower bound is sharp. In order to derive these results we first prove the following lemma in Chapter 6 and deduce two other results which we list here without proofs.

Lemma 1.3.5. *There is no common inner edge between any two different subgraphs of G in $\Re$.*

The proof of the main result in Chapter 6 is by induction, and is based on the following two results.

Theorem 1.3.12 *Let G be a 4-connected graph and F a maximal l-bi-fan of G with $l \geq 2$. Then there exists an edge e' in F such that $e' \in E_R(G)$ and $e_R(G) \geq e_R(G \ominus e') + 1$.*

Theorem 1.3.13 *Let G be a 4-connected graph and L a maximal l-belt of G with $l \geq 3$. Then there exists an edge e' in $E(G)$ such that $e_R(G) \geq e_R(G \ominus e') + 2$.*

In the next paragraph we are going to list another result obtained in Chapter 6. Before we can formulate this result we need a definition.

A 4-connected graph G is said to have property $(\star)$ if there does not exist any edge $xy \in E_R(G)$ such that both $d(x) \geq 5$ and $d(y) \geq 5$.

If the subgraph of a 4-connected graph G induced by $E_N(G)$ is a forest, then it is easy to get the bound of the number of removable edges of G; if a 4-connected graph G contains a cycle C such that $E(C) \subset E_N(G)$, then we have the following result holds.

Theorem 1.3.14 *Let G be a 4-connected graph with property $(\star)$, $|G| \geq 8$, and let C' be a cycle of G. If C' does not contain any removable edges of G, then G has one of the following structures as subgraph: l-belt, l-bi-fan $(l \geq 1)$, W-framework, W'-framework or helm, such that the subgraph intersects C' at some of its inner edge(s).*

The following three results are used in the proof of our main results.

Theorem 1.3.15 *Let G be a 4-connected graph with property $(\star)$. Suppose, H is a helm of G as in Definition 1.2.1. Let $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and let $P = y_1y_2 \cdots y_h$ be a path in $[E_N(G)]$ with $h \geq 2$ such that $a \notin V(P)$ and $\{y_1, y_h\} \subset \{x_1, x_2, x_3, x_4\}$. Then G contains one of the following structures H_1 as its subgraph: l-belt, l-bi-fan, $(l \geq 1)$, W-framework, W'-framework or helm, such that at least one inner edge of H_1 belongs to $E(P \cup H)$, and H and H_1 do not have any common inner edge.*

Theorem 1.3.16 *Let G be a 4-connected graph with property $(\star)$ and L_1 a maximal 1-belt of G as in Definition 1.2.3 such that $V(L_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Suppose that $P = l_1 l_2 \cdots l_h$ is a path of $[E_N(G)]$ such that $\{l_1, l_h\} \subset \{x_1, x_3, y_1, y_3\}$ and $\{x_2, y_2\} \cap V(P) = \emptyset$. Then G contains one of the following structures L' as subgraph: l-belt, $(l \geq 1)$, helm, W-framework, W'-framework or l-bi-fan, $(l \geq 1)$, such that at least one inner edge of L' belongs to $E(P \cup L_1)$.*

Theorem 1.3.17 *Let G be a 4-connected graph with property $(\star)$ and L_1' a maximal 1-co-belt of G as in Definition 1.2.4 with $V(L_1') = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}$. Suppose that $P = l_1 l_2 \cdots l_h$ is a path of $[E_N(G)]$ such that $\{x_2, x_3, y_2\} \cap V(P) = \emptyset$ and $\{l_1, l_h\} \subset \{x_1, x_4, y_1, y_3\}$. Then, G contains one of the following structures as subgraph: l-belt, $(l \geq 1)$, W-framework, W'-framework, helm or l-bi-fan, $(l \geq 1)$, such that it has some inner edge(s) belonging to $E(P)$.*

In the following we describe the construction of graphs for which our lower bound is sharp:

Let M be a 5-wheel such that $V(M) = \{a, x, y, z, v\}$ and a is its center. Let T_1, T_2, T_3, T_4 be four trees such that for each $i \in \{1, 2, 3, 4\}$, T_i has k vertices of degree one and $|T_i| - k$ vertices of degree four. Let the vertices of degree four be ${u_i}^{(1)}, {u_i}^{(2)}, \cdots, {u_i}^{(|T_i|-k)}$, and the vertices of degree one be ${x_i}^{(1)}, {x_i}^{(2)}, \cdots, {x_i}^{(k)}$. Let $M_1, M_2, \cdots, M_k$ be k copies of M and $a^{(j)}, x^{(j)}, y^{(j)}, z^{(j)}, v^{(j)}$ the vertices of M_j corresponding to the vertices a, x, y, z, v of M, respectively, where $j = 1, 2, \cdots, k$. For each $j \in \{1, \cdots, k\}$, identify ${x_1}^{(j)}, {x_2}^{(j)}, {x_3}^{(j)}, {x_4}^{(j)}$ with $x^{(j)}, y^{(j)}, z^{(j)}, v^{(j)}$ such that each of ${x_1}^{(j)}, {x_2}^{(j)}, {x_3}^{(j)}, {x_4}^{(j)}$ corresponds to one and only one of $x^{(j)}, y^{(j)}, z^{(j)}, v^{(j)}$. Denote the resulting graph by G. It is easy to see that G is 4-connected.

Next we will show that for each 4-cycle $C = x^{(j)} y^{(j)} z^{(j)} v^{(j)} x^{(j)}$ of G, we have that $E(C) \subset E_R(G)$, and the other edges in G are unremovable, where $j = 1, 2, \cdots, k$. For $y^{(j)} {u_i}^{(l)} \in E(G)$, let $S = \{x^{(j)}, v^{(j)}, z^{(j)}\}, A = \{a^{(j)}, y^{(j)}\}, B = G - y^{(j)} {u_i}^{(l)} - S - A$. Then $(y^{(j)} {u_i}^{(l)}, S; A, B)$ is a separating group of G, and hence $y^{(j)} {u_i}^{(l)} \in E_N(G)$. By symmetry, we can show that $x^{(j)}{u_i}^{(l)}, z^{(j)}{u_i}^{(l)}, v^{(j)}{u_i}^{(l)} \in E_N(G)$, where $j = 1, 2, \cdots, k; i = 1, 2, 3, 4; l = 1, 2, \cdots, |T| - k$. For each edge $a^{(j)}x^{(j)}$, it is easy to see that $(a^{(j)}x^{(j)}, T)$ is a separating pair of G such that $T = \{y^{(j)}, v^{(j)}, {u_i}^{(j)}\}$ and ${u_i}^{(l)}z^{(j)} \in E(G)$. By symmetry, $a^{(j)}y^{(j)}, a^{(j)}z^{(j)}, a^{(j)}v^{(j)} \in E_N(G)$. Using Corollary 1.3.1 it is easy to see that for each 4-cycle $C = x^{(j)}y^{(j)}z^{(j)}v^{(j)}x^{(j)}$, we have $E(C) \subset E_R(G)$. For each edge e of T_i, for example, $e = {u_1}^{(l)}{u_1}^{(l+1)}$, it is easy to see that (e, S) is a separating pair of G such that $S = \{{u_2}^{(l)}, {u_3}^{(l)}, {u_4}^{(l)}\}$. Therefore, for each edge e of T_i, where $i = 1, 2, 3, 4$, we have that $e \in E_N(G)$, and so $e_R(G) = 4k, |T_i| = (3k-2)/2, (i = 1, 2, 3, 4), |G| = 7k - 4, e_R(G) = (4|G| + 16)/7$. We denote the set of all these constructed graphs by $\Im$. See Figure 1.11.

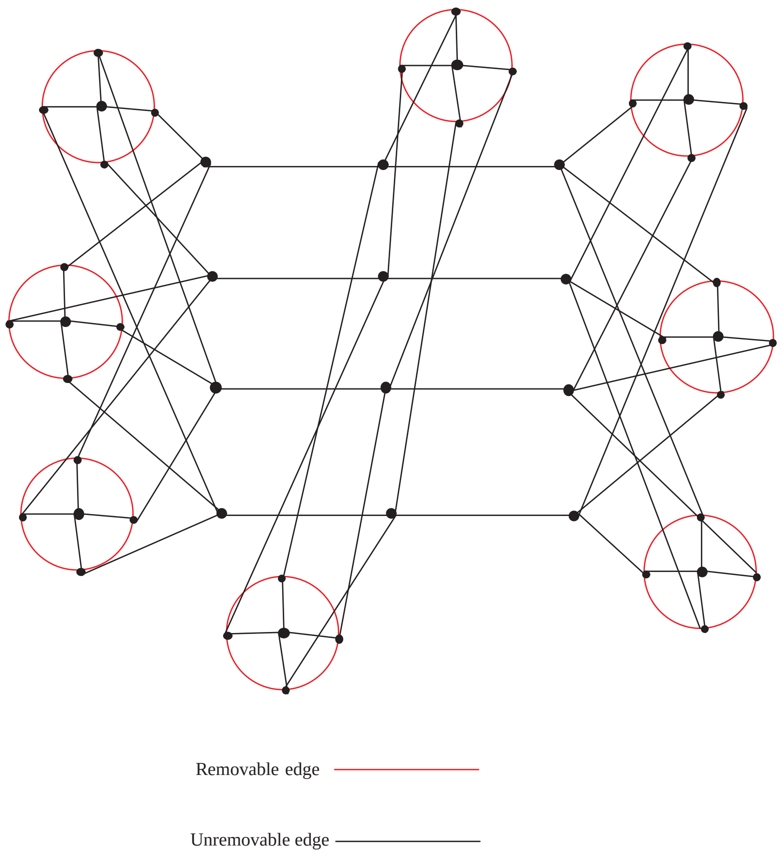

Figure 1.11:

Now our main result of which proof can be found in Chapter 6 is as follows:

Theorem 1.3.18. *Let G be a 4-connected graph of order at least 5. If G is neither C_5^2 nor C_6^2, then $e_R(G) \geq (4|G|+16)/7$ and the equality holds if and only if $G \in \Im$.*

In addition, we study the distribution of removable edges in a spanning tree or outside a cycle in a 4-connected graph in Chapter 7. We obtain the following results.

Theorem 1.3.19 *Let G be a 4-connected graph which does not contain any subgraph belonging to $\Re$. Then any spanning tree T of G contains at least one removable edge.*

We can give an example to show that the above result cannot be improved.

For the distribution of removable edges outside a cycle in a 4-connected graph, we present the following results together with examples that show the results are in some sense best possible.

Theorem 1.3.20 *Let G be a 4-connected graph and C a cycle of G. If C does not pass through any l-belt or l-co-belt, then there are at least two removable edges outside C.*

Theorem 1.3.21 *Let G be a 4-connected graph and C a cycle of G. If C passes through only one l-belt or l-co-belt, then there is at least one removable edge outside C.*

Chapter 2

Removable edges in 4-connected graphs and the structure of 4-connected graphs

In this chapter we introduce some results which were obtained by Yin. Since those results are published in Chinese, for convenience, we present them here together with their proofs. We use some new ideas in some of the proofs of the results.

2.1 Some results and their proofs

The following results on the properties of removable edges in 4-connected graphs will be used frequently in this dissertation, but were obtained by Yin in [43].

First, we list the following result which holds clearly, so we omit its proof.

Theorem 2.1.1. *Let G be a 4-connected graph with $|G| \geq 7$. An edge e of G is unremovable if and only if there is a separating pair (e, S), or a separating group $(e, S; A, B)$ in G.*

Theorem 2.1.2. *Let G be a 4-connected graph with $|G| \geq 8$ and $(xy, S; A, B)$ a separating group of G such that $x \in A$, $y \in B$ and $|A| \geq 3$. Then every edge* *in* $[\{x\}, S]$ *is removable.*

Proof. By contradiction. Let $e = xu$ such that $u \in S$. And suppose $xu \in E_N(G)$. Consider the corresponding separating group $(xu, T; C, D)$ such that $x \in C, u \in D$. Let

$$X_1 = (C \cap S) \cup (S \cap T) \cup (A \cap T)$$

$$X_2 = (A \cap T) \cup (S \cap T) \cup (S \cap D)$$

$$X_3 = (D \cap S) \cup (S \cap T) \cup (B \cap T)$$

$$X_4 = (B \cap T) \cup (S \cap T) \cup (C \cap S)$$

We distinguish two cases to complete the proof.

Case 1. $y \in B \cap C$.

Since $y \in B \cap C$, X_4 is a vertex-cut of $G - xy$. Since G is a 4-connected graph, $|X_4| \geq 3$. Noticing that $|X_2| + |X_4| = |S| + |T| = 6$, we get $|X_2| \leq 3$, so $A \cap D = \emptyset$. We claim that $B \cap D \neq \emptyset$: otherwise, $B \cap D = \emptyset$, implying $|D \cap S| \geq 2$, so $|S \cap (C \cup T)| \leq 1$, and hence $|B \cap T| \geq 2$. From $|T| = 3$ we can get that $|A \cap T| \leq 1$, and so $|X_1| \leq 2$. Since $|A| \geq 3$, we have that $|A \cap C| \geq 2$, But now it can be checked easily that $\{x\} \cup X_1$ is a vertex-cut of G with cardinality less than 4, a contradiction. So $B \cap D \neq \emptyset$, and $|X_3| \geq 4$. From $|X_1| + |X_3| = 6$ we get $|X_1| \leq 2$. By a similar argument as before this implies $\{x\} \cup X_1$ is a vertex-cut of G with cardinality less than 4, a contradiction. Therefore, Case 1. does not occur.

Case 2. $y \in B \cap T$.

First, we claim that $A \cap C = \{x\}$. Otherwise, suppose $|A \cap C| \geq 2$. We first claim that $|X_1| \geq 3$: otherwise, $|X_1| \leq 2$, and then $\{x\} \cup X_1$ is a vertex-cut of G with cardinality less than 4, a contradiction. So $|X_1| \geq 3$, and so $|X_3| \leq 3$, which implies that $B \cap D = \emptyset$. First consider the case that $A \cap D = \emptyset$. Then $|D \cap S| \geq 2$ and $|S \cap (C \cup T)| \leq 1$. Since $|X_1| \geq 3$, $|A \cap T| \geq 2$, and hence $|B \cap T| = 1$. Now it can be checked easily that $|X_4| \leq 2$, hence $B \cap C = \emptyset$, and so $|B \cap T| = 1$, which contradicts $|B| \geq 2$. Consequently we may assume $A \cap D \neq \emptyset$, and so $|X_2| \geq 4$. By symmetry, we can assume that $B \cap C \neq \emptyset$, and so $|X_4| \geq 4$. But now $|X_2| + |X_4| \geq 8$, which contradicts that $|X_2|+|X_4| = |S|+|T| = 6$. This contradiction confirms that $A \cap C = \{x\}$. Since A and B are connected subgraphs of G, we have that $A \cap T \neq \emptyset, C \cap S \neq \emptyset$. If $S \cap T = \emptyset$, it can be checked easily that $|C \cap S| = |A \cap T| = 1$ and $|B \cap T| = |D \cap S| = 2$. Then we have that $A \cap D = \emptyset = B \cap C$. This implies that $|A| = |C| = 2$, which contradicts $|A| \geq 3$. So, $S \cap T \neq \emptyset$, Then $|C \cap S| = |A \cap T| = |S \cap T| = |B \cap T| = |D \cap S| = 1$. It is easy to see that $|X_2| = |X_4| = 3$, so $A \cap D = \emptyset = B \cap C$, which contradicts $|A| \geq 3$. This complete the proof of Theorem 2.1.2.$\square$

Corollary 2.1.3. *Let G be a 4-connected graph with $|G| \geq 8$. Then every 3-cycle of G contains at least one removable edge.*

Theorem 2.1.4. *Let G be a 4-connected graph with $|G| \geq 7$. If for an unremovable edge xy, i.e., $xy \in E_N(G)$, there is a separating group $(xy, S; A, B)$, then all the edges in $E([S])$ are removable.*

Proof. Let $ab \in E_N(G) \cap E([S])$, and consider the corresponding separating group $(ab, T; C, D)$ such that $a \in C, b \in D$. Let X_1, X_2, X_3, X_4 be defined as in the proof of Theorem 2.1.2. We distinguish the following cases to complete the proof.

Case 1. $x \in A \cap C$.

We deal with the following subcases separately.

Subcase 1.1. $y \in B \cap C$.

Since X_1 is a vertex-cut of $G-xy$, $|X_1| \geq 3$. So $|X_3| \leq 3$, and so $B \cap D = \emptyset$. Similar arguments yield $A \cap D = \emptyset$. Since $D = D \cap S$, $|D \cap S| \geq 2$, which implies that $|D \cap S| = 2$ and $C \cap S = \{a\}$. It is easy to see that $S \cap T = \emptyset$. Since $|X_1| \geq 3$, $|A \cap T| \geq 2$. But then $|X_4| \leq 2$, a contradiction.

Subcase 1.2. $y \in B \cap T$.

From $|X_1| \geq 3$ we get that $|X_3| \leq 3$, hence $B \cap D = \emptyset$. If $A \cap T = \emptyset$, since A is a connected subgraph of G, $A \cap D = \emptyset$. Then we have $|A \cap C| \geq 2$, and $|S \cap D| \geq 2$, which contradicts $|X_1| \geq 3$. So $A \cap T \neq \emptyset$. If $A \cap D \neq \emptyset$, then $|X_2| \geq 4$. Since $|X_2| + |X_4| = 6$, we get $|X_4| \leq 2$, and so $B \cap C = \emptyset$, which implies $|B \cap T| \geq 2$. Now it is checked readily that $|X_2| \leq 3$, which contradicts $|X_2| \geq 4$. So $A \cap D = \emptyset$. Then $|S \cap D| = 2$ and $S \cap T = \emptyset$. Since $|X_1| \geq 3$, we get $|A \cap T| = 2$ and $B \cap T = \{y\}$. Hence $|X_4| = 2$, so $B \cap C = \emptyset$, and $B = B \cap T = \{y\}$, which contradicts $|B| \geq 2$.

Case 2. $x \in A \cap T$.

We deal with the following subcases:

Subcase 2.1. $y \in B \cap T$.

We claim that $S \cap T = \emptyset$. Otherwise, we have $|X_1| = |X_2| = |X_3| = |X_4| = 3$. Since G is 4-connected graph, we have $A \cap C = \emptyset = A \cap D$ and $B \cap C = \emptyset = B \cap D$, a contradiction. By symmetry, we may assume that $|A \cap T| = 2$, so either $|C \cap S| = 2$ or $|D \cap S| = 2$. Without loss of generality, we may assume $|C \cap S| = 2$, then we will have that $A \cap D = \emptyset = B \cap D$, and $D = S \cap D = \{b\}$, which contradicts $|D| \geq 2$.

Subcase 2.2. $y \in B \cap C$.

By symmetry, we may treat this as in Subcase 1.2 of Case 1.

Subcase 2.3. $y \in B \cap D$.

By symmetry, we may treat this as in Subcase 1.2 of Case 1.

Case 3. $x \in A \cap D$.

We may treat this as in Case 1.

The proof of Theorem 2.1.4 is complete. □

2.2 A Characterization of 4-Connected Graphs

Here we present the method due to Yin [43] for constructing all 4-connected graphs from the 2-cyclic graphs. Since we do not use those results in the other chapter, we omit the proof of the following results.

Lemma 2.2.1. *Let G be a 4-connected graph with $|G| \geq 7$, and let $z \in V(G)$ such that $d(z) \geq 6$. We split vertex z into two vertices x and y, and join x to y. Then the neighbors of z to either x or y in such a way that $d(x) \geq 4, d(y) \geq 4$. Then the resultant graph G' is 4-connected.*

The above operation from G to G' in Lemma 2.2.1 is called *vertex splitting*.

Let F_1 denote the graph with $V(F_1) = \{x_1, x_2, x_3, x\}$ and $E(F_1) = \{x_1x_2, x_2x_3, x_3x_1, xx_1\}$.

Lemma 2.2.2. *Let G be a 4-connected graph, and suppose G contains F_1 as a subgraph. We add one vertex y and four edges yx_1, yx_2, yx_3, yx to G. If we delete any of the edges $\{x_1x_2, x_1x_3, x_2x_3\}$ in such a way that in the new graph G' all vertices have degree at least four, then G' is 4-connected.*

The above operations from G to G' are called *F_1-operations*.

Let F_2 denote the graph with $V(F_2) = \{z, a_1, a_2, a_3, a_4, a_5, a_6\}$ and $E(F_2) =$

$\{za_1, za_2, za_3, za_4, za_5, za_6, a_1a_2, a_3a_4, a_5a_6\}$.

Lemma 2.2.3. *Let G be a 4-connected graph with $|G| \geq 7$, and suppose G contains F_2 as a subgraph with $d_G(z) = 6$. We extend vertex z into a 3-cycle $z'xyz'$, join x to vertices a_1, a_2, y, z', y to vertices a_3, a_4, x, z' and z' to vertices a_5, a_6, x, y. The new graph G' is 4-connected.*

The above operation from G to G' is called *vertex extension.*

Theorem 2.2.5. *G is a 4-connected graph if and only if either G is a 2-cyclic graph or G can be obtained from a 2-cyclic graph by applying the following four operations: (i) adding edges, (ii) vertex splitting, (iii) F_1-operation and (iv) vertex extension.*

Chapter 3

Removable Edges in a Cycle of a 4-Connected Graph

In this chapter we investigate how many removable edges there are in a cycle of a 4-connected graph, and give examples to show that our results are in some sense best possible.

3.1 Some Preliminary Results

In this chapter we shall obtain lower bounds on the number of removable edges in a cycle of a 4-connected graph. Before we present and can prove our main results, we need to prove some lemmas. The following lemma is a key ingredient for the proof of our main results.

Lemma 3.1.1. *Let G be a 4-connected graph, $(xy, S; A, B)$ be a separating group of G such that $x \in A, y \in B, S = \{a, b, c\}$ and A be a 1-edge-vertex atom, say $A = \{x, z\}$. Then one of the following conclusions holds:*

(i) $ax, bx, zx \in E_R(G)$.

(ii) $ax \in E_N(G), d(x) = d(z) = 4, bx, zx, az \in E_R(G), zc \in E_N(G)$.

(iii) $ax \in E_N(G), ay \in E_R(G)$. *Moreover, if* $d(a) = 4, d(y) \geq 5$, *then* $az, zb, zx, by \in E_R(G), bx \in E_N(G)$*; if* $d(a) \geq 5, d(y) = 4$, *then* $by, bx, bz, az \in E_R(G), zx \in E_N(G)$, *if* $d(a) = d(y) = 4$, *then* $az, bz, by \in E_R(G), bx, zx \in$

$E_N(G)$, if $d(a) \geq 5, d(y) \geq 5$, then $az, zx, bx, by \in E_R(G)$.

(iv) *$ax, bx, ac, bc \in E_R(G), zx, zc \in E_N(G), \{za, zb\} \cap E_N(G) \neq \emptyset, d(x) = d(c) = d(z) = 4$. If $za \in E_N(G)$, then the following conclusion holds: $d(b) = 4$, and if $d(a) = 4$, then $bz \in E_N(G)$; if $d(a) \geq 5$, then $bz \in E_R(G)$ holds. If $bz \in E_N(G)$, then the following conclusion holds: $d(a) = 4$, and if $d(b) = 4$, then $az \in E_N(G)$; if $d(b) \geq 5$, then $az \in E_R(G)$.*

(v) *$ax, bx, az, bz \in E_R(G), xz \in E_N(G), d(x) = d(z) = 4$.*

(vi) *$bx \in E_N(G), by \in E_R(G)$. Moreover, if $d(b) = 4, d(y) \geq 5$, then $bz, za, zx, ay \in E_R(G), ax \in E_N(G)$; if $d(b) \geq 5, d(y) = 4$, then $ay, ax, az, bz \in E_R(G), zx \in E_N(G)$, if $d(b) = d(y) = 4$, then $bz, az, ay \in E_R(G), ax, zx \in E_N(G)$, if $d(b) \geq 5, d(y) \geq 5$, then $bz, zx, ax, ay \in E_R(G)$.*

(vii) *$bx \in E_N(G), d(x) = d(z) = 4, ax, zx, bz \in E_R(G), zc \in E_N(G)$.*

Proof. If $ax, bx, zx \in E_R(G)$, then conclusion (i) holds. So, we may assume that $\{ax, bx, zx\} \cap E_N(G) \neq \emptyset$. Next we will distinguish the following cases to complete the proof.

Case 1. $ax \in E_N(G)$.

Then we consider the corresponding separating group $(ax, T; C, D)$ such that $x \in C, a \in D$, and so, $x \in A \cap C, y \in B \cap (C \cup T)$. Let

$$X_1 = (C \cap S) \cup (S \cap T) \cup (A \cap T)$$

$$X_2 = (A \cap T) \cup (S \cap T) \cup (S \cap D)$$

$$X_3 = (D \cap S) \cup (S \cap T) \cup (B \cap T)$$

$$X_4 = (B \cap T) \cup (S \cap T) \cup (C \cap S)$$

We distinguish a number of subcases.

Subcase 1.1. $y \in B \cap C$.

Since $|A| = 2$ and A is a connected subgraph of G, we have $A \cap D = \emptyset$. First, we claim that $A \cap T \neq \emptyset$. Otherwise, $A \cap T = \emptyset$, and so $|A \cap C| = 2$. Since $a \in S \cap D$, we have $|X_1| \leq 2$. Then $X_1 \cup \{x\}$ is a vertex-cut of G with cardinality less than 4, a contradiction. Hence, $A \cap T = \{z\}$. Second, we claim that $S \cap T = \emptyset$. Otherwise, $S \cap T \neq \emptyset$, and a contradiction will be deduced as follows: If $B \cap T = \emptyset$, since B is a connected subgraph of G, we have $B \cap D = \emptyset$. Then $B = B \cap C$, and so $|S \cap T| = 2$. Noticing that $a \in S \cap D$ and $|S| = 3$, we have $S \cap C = \emptyset$. Since $|B| \geq 2$, we know that $|B \cap C| \geq 2$. Then it is easy to see that $\{y\} \cup (S \cap T)$ is a vertex-cut of G with cardinality less than 4, a contradiction. So, $B \cap T \neq \emptyset$, and so $|S \cap T| = 1$. Noticing that $|T| = 3$, we have $|B \cap T| = 1$. Since X_4 is a vertex-cut of $G - xy$, we have $|X_4| \geq 3$, and so $|S \cap C| \geq 1$. Since $S \cap D \neq \emptyset$, by noticing that $|S| = 3$, we have $|S \cap D| = 1$, i.e., $S \cap D = \{a\}$. Note that $|X_3| = 3$. Since G is 4-connected, we have $B \cap D = \emptyset$. Hence, $D = \{a\}$, which contradicts $|D| \geq 2$. Therefore, $S \cap T = \emptyset$. Note that $|B \cap T| = 2$. If $|S \cap D| = 1$, by similar arguments we can get that $D = \{a\}$, a contradiction. So, $|S \cap D| \geq 2$. Since $|X_4| \geq 3$, we have $|S \cap C| \geq 1$. Therefore, $|S \cap C| = 1$ and $|S \cap D| = 2$. Since $bx \in E(G)$, obviously we have $b \in X_1$, and so $S \cap C = \{b\}$. Then $S \cap D = \{a, c\}, \Gamma_G(x) = \{a, b, y, z\}, \Gamma_G(z) = \{x, a, b, c\}$. We claim that $xz \in E_R(G)$. Otherwise, $xz \in E_N(G)$, and we consider the corresponding separating group $(xz, S'; A', B')$ such that $x \in A', z \in B'$. Since $xzax$ is a 3-cycle of G, we have that $a \in S'$ and $ax \in E_N(G)$. By Theorem 2.1.2 we know that $|A'| = 2$, say $A' = \{x, v_1\}$. Then we have that axv_1a is a 3-cycle of G and $v_1 \neq z$, which is impossible, and so $xz \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we consider the corresponding separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $x \in S'$. Since $ax \in E_N(G)$, by Theorem 2.1.2 we have $|A'| = 2$, say $A' = \{a, v_1\}$. Then axv_1a is a 3-cycle of G and $v_1 \neq z$, which is impossible, and so $az \in E_R(G)$.

Let $S' = \{x\} \cup (B \cap T), A' = C \cap (B \cup S), B' = G - bz - S' - A'$. Then $(bz, S'; A', B')$ is a separating group of G, and so $bz \in E_N(G)$. We claim that $bx \in E_R(G)$. Otherwise, $bx \in E_N(G)$, and we consider the corresponding separating group $(bx, S'; A', B')$ such that $b \in A', x \in B'$. Since $bxzb$ is a 3-cycle of G, we have $z \in S'$. Since $bz \in E_N(G)$, we have $|A'| = 2$, say $A' = \{b, v_1\}$. Then bv_1zb is a 3-cycle of G, and $v_1 \neq x$, which is impossible. Hence $bx \in E_R(G)$. Let $S_1 = \{a, b, y\}$, then (zc, S_1) is a separating pair of G, and so $zc \in E_N(G)$. Obviously, $d(x) = d(z) = 4$. Hence, conclusion (ii) holds.

Subcase 1.2. $y \in B \cap T$.

Since $xy \in E_N(G)$, by Theorem 2.1.2 we have $|C| = 2$. If $|A \cap C| = 2$, then we have $A = A \cap C = C$. Since $B \cap T \neq \emptyset \neq S \cap D$, we have $|S \cap T| \leq 2$. It is easy to see that $\{x\} \cup X_1$ is a vertex-cut of G with cardinality less than 4, a contradiction. So $A \cap C = \{x\}$. Since A and C are connected subgraphs of G, we have that $|S \cap C| = |A \cap T| = 1$ and $B \cap C = \emptyset = A \cap D$. We claim that $S \cap T = \emptyset$. Otherwise, $|S \cap T| = 1$, and so $|B \cap T| = 1$. Note that $|X_3| = 3$. Since G is 4-connected, we have $B \cap D = \emptyset$, and so $B = B \cap T = \{y\}$, which contradicts $|B| \geq 2$. Therefore, $S \cap T = \emptyset$, and so $|B \cap T| = |S \cap D| = 2$. From $\Gamma_G(x) = \{z, b, a, y\}$ we know that $S \cap C = \{b\}$, and so $S \cap D = \{a, c\}, A \cap T = \{z\}$. Let $B \cap T = \{u, y\}$. Next we will deal with the following subcases.

Subcase 1.2.1. $ay \notin E(G)$.

We claim that $xz \in E_R(G)$. Otherwise, $xz \in E_N(G)$. We consider the corresponding separating group $(xz, S'; A', B')$ such that $z \in A', x \in B'$. Since $azxa$ is a 3-cycle of G, we have $a \in S'$. Since $ax \in E_N(G)$, by Theorem 2.1.2 we have that $|B'| = 2$, say $B' = \{x, v_1\}$. Then axv_1a is a 3-cycle of G. However, $ay \notin E(G)$ and $v_1 \neq z$, which is impossible. Hence, $xz \in E_R(G)$. By symmetry, $bx \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$. We consider the corresponding separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Since $azxa$ is a 3-cycle of G, we have $x \in S'$. Since $ax \in E_N(G)$, we have that $|A'| = 2$, say $A' = \{a, v_1\}$.

Then axv_1a is a 3-cycle of G, and analogous arguments yield a contradiction. So $az \in E_R(G)$. By symmetry, $by \in E_R(G)$. Let $S' = \{a, b, y\}$. Obviously, (zc, S') is a separating pair of G, and so $zc \in E_N(G)$. Hence, conclusion (ii) holds.

Subcase 1.2.2. $ay \in E(G)$.

Then by Corollary 2.1.3 we know that $ay \in E_R(G)$. Then, we consider the following cases.

(1.) $d(a) \geq 5$ and $d(y) \geq 5$. We claim that $xz \in E_R(G)$. Otherwise, $xz \in E_N(G)$, and we consider the corresponding separating group $(xz, S'; A', B')$ such that $x \in A', z \in B'$. Since $azxa$ is a 3-cycle of G, we have $a \in S'$. Since $ax \in E_N(G)$, by Theorem 2.1.2 we know that $|A'| = 2$, say $A' = \{x, v_1\}$. Then axv_1a is a 3-cycle of G. Noticing that $d(v_1) = 4$ and $d(y) \geq 5$, we have that $v_1 \neq y$, which is impossible. Hence, $xz \in E_R(G)$. By symmetry, $bx \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we consider the corresponding separating group $(az, S'; A', B')$. Obviously, $x \in S'$, and analogous arguments yield to a contradiction. So, $az \in E_R(G)$. By symmetry, $by \in E_R(G)$. Hence, conclusion (iii) holds.

(2.) $d(a) = 4$ and $d(y) \geq 5$. We let $\Gamma_G(a) = \{x, y, z, v\}$. Let $A' = \{a, x\}, S' = \{v, z, y\}, B' = G - bx - S' - A'$. Then $(bx, S'; A', B')$ is a separating group of G, and so $bx \in E_N(G)$. We claim that $bz \in E_R(G)$. Otherwise, $bz \in E_N(G)$. We consider the corresponding separating group $(bz, S'; A', B')$ such that $b \in A', z \in B'$. Noticing that $bzxb$ is a 3-cycle of G, we have $x \in S'$. Since $bx \in E_N(G)$, from Theorem 2.1.2 we have that $|A'| = 2$, say $A' = \{b, v_1\}$. Then bxv_1b is a 3-cycle of G. Noticing that $d(y) \geq 5$ and $d(v_1) = 4$, we have that $v_1 \neq y$, which is impossible. Therefore, $bz \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$. We consider the separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $x \in S'$. Since $ax \in E_N(G)$, from Theorem 2.1.2 we have that $|A'| = 2$, say $A' = \{a, v_1\}$. Then axv_1a is a 3-cycle of G and $v_1 \neq z$. Note that $d(v_1) = 4, d(y) \geq 5$, and so $v_1 \neq y$, which is impossible. So $az \in E_R(G)$. By analogous arguments we can show that

$zx \in E_R(G)$. We claim that $by \in E_R(G)$. Otherwise, $by \in E_N(G)$. We consider the separating group $(by, S'; A', B')$ such that $b \in A', y \in B'$. Obviously, $x \in S'$. Since $xy \in E_N(G)$, from Theorem 2.1.2 we have that $|B'| = 2$, say $B' = \{y, v_1\}$. Then xyv_1x is a 3-cycle of G. It is easy to see that this is true only if $v_1 = a$. From $\Gamma_G(a) = \{x, y, z, v\}$ we know that $S' = \{x, z, v\}$. Since $d(y) \geq 5$, we have $yz \in E(G)$, which is impossible. So $by \in E_R(G)$. Hence, conclusion (iii) holds.

(3.) $d(a) \geq 5$ and $d(y) = 4$. By analogous arguments as used in (2.) we can show that the conclusion (iii) holds.

(4.) $d(a) = d(y) = 4$. We let $\Gamma_G(a) = \{x, y, z, v\}$, $A_1 = \{a, x\}, S_1 = \{z, y, v\}, B_1 = G - bx - S_1 - A_1$. Then $(bx, S_1; A_1, B_1)$ is a separating group of G, and so $bx \in E_N(G)$. By symmetry, $ax, xy, zx \in E_N(G)$. From Corollary 2.1.3 we have that $az, by, bz \in E_R(G)$. Hence, the conclusion (iii) holds.

If $bx \in E_N(G)$, we can apply similar arguments to show that conclusion (vi) or (vii) hold. So, next we may assume that $ax, bx \in E_R(G)$.

Case 2. $xz \in E_N(G)$.

We consider the corresponding separating group $(xz, T; C, D)$ such that $x \in C, z \in D$. Then $x \in A \cap C, z \in A \cap D$. Since $xzax, xzbx$ are two 3-cycles of G, we have that $a, b \in S \cap T$. Since $A \cap D = \{z\}$ and D is a connected subgraph of G with $|D| \geq 2$, we get that $S \cap D \neq \emptyset$. Since $S = \{a, b, c\}$, we have that $S \cap D = \{c\}$. Obviously, $|B \cap T| = 1$. We distinguish three subcases.

Subcase 2.1. $az \in E_N(G)$.

By Theorem 2.1.2 we have that $|D| = 2$, and so $D = \{z, c\}$. It is easy to see that $ac, bc \in E(G)$. From Theorem 2.1.4 we have that $ac, bc \in E_R(G)$. Obviously, $d(x) = d(c) = d(z) = 4$ and $\Gamma_G(x) = \{z, b, a, y\}$. Let $A_1 = \{x, z\}, S_1 = \{y, a, b\}, B_1 = G - zc - S_1 - A_1$ Then $(zc, S_1; A_1, B_1)$ is a separating group of G, and so $zc \in E_N(G)$. We take the separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $x \in S'$. Since $xz \in E_N(G)$, we have that $|B'| = 2$, say $B' = \{z, v_1\}$. Then xzv_1x is a 3-cycle of G, which is true only if $v_1 = b$, and so $d(b) = 4$. Now if $d(a) = 4$, let $\Gamma_G(a) = \{x, z, c, v\}$, $A_1 = \{a, z\}, S_1 = \{c, x, v\}$ and $B_1 = G - bz - S_1 - B_1$. Then $(bz, S_1; A_1, B_1)$ is a separating group of G, and so $bz \in E_N(G)$. If $d(a) \geq 5$, we claim that $bz \in E_R(G)$. Otherwise, $bz \in E_N(G)$. Then we consider the corresponding separating group $(bz, S_1; A_1, B_1)$ such that $b \in A_1, z \in B_1$. Obviously, $x \in S_1$. Since $xz \in E_N(G)$, from Theorem 2.1.2 we have $|B_1| = 2$, say $B_1 = \{z, v_1\}$. Then xv_1zx is a 3-cycle of G. Note that $d(a) \geq 5, d(v_1) = 4$, and so $v_1 \neq a$, which is impossible. So, $bz \in E_R(G)$. Hence, conclusion (iv) holds.

Subcase 2.2. $bz \in E_N(G)$.

We can apply similar arguments as used in Subcase 2.1 to show that conclusion (iv) holds.

Subcase 2.3 $az, bz \in E_R(G)$.

Obviously, $d(x) = d(z) = 4$, and so conclusion (v) holds. This completes the proof.□

From Lemma 3.1.1 and its proof, we deduce the following corollary.

Corollary 3.1.1. *Let G be a 4-connected graph and $(xy, S; A, B)$ be a separating group of G such that $x \in A, y \in B, S = \{a, b, c\}$. Let A be a 1-edge-vertex-cut atom, say $A = \{x, z\}$. If $\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$, then x is an inner vertex of one of the following subgraphs in G: helm, l-co-belt, l-belt, W'-framework, W-framework or l-bi-fan.*

The following lemma will be used in the proof of Theorem 3.2.1.

Lemma 3.1.2. *Let G be a 4-connected graph, $(xy, S; A, B)$ be a separating group of G, and A be a 2-edge-vertex-cut atom, say $A = \{x, z\}$ and $S = \{a, b, c\}$. Then $ax, bx, cx, xz \in E_R(G)$.*

Proof. By contradiction. Assuming at least one of the edges ax, bx, cx, xz belong to $E_N(G)$. We consider the following cases.

Case 1. $ax \in E_N(G)$.

We consider the corresponding separating group $(ax, T; C, D)$ such that $x \in C, a \in D$. Then $x \in A \cap C, a \in S \cap D$. Let $X = (D \cap S) \cup (S \cap T) \cup (B \cap T)$. Since $bx, cx \in E(G)$, we get that $b, c \in S \cap (C \cup T)$, and so $|S \cap D| = 1$. We claim that $A \cap T \neq \emptyset$. Otherwise, $A \cap T = \emptyset$. Since $|A| = 2$ and A is a connected subgraph of G, we have that $A \cap C = \{x, z\}$. It is easy to see that $\{b, c, x\}$ is a 3-vertex-cut of G, a contradiction. Therefore, $A \cap T = \{z\}, A \cap D = \emptyset$. Obviously, $|X| \geq 3$. Since $|S \cap D| = 1$ and $|D| \geq 2$, we have that $B \cap D \neq \emptyset$, and so $|X| \geq 4$. However, by noticing that $|A \cap T| = 1$, we find $|(S \cup B) \cap T| = 2$, and then $|X| = 3$, a contradiction.

If $bx \in E_N(G)$ or $cx \in E_N(G)$, we can apply similar arguments. So, next we may assume that $bx, cx \in E_R(G)$.

Case 2. $xz \in E_N(G)$.

We consider the corresponding separating group $(xz, T; C, D)$ such that $x \in C, z \in D$. Then we have $x \in A \cap C, z \in A \cap D$. It is easy to see that $a, b, c \in S \cap T$. Since $|T| = 3$, we have that $y \in B \cap C$. Let $X = (D \cap S) \cup (S \cap T) \cup (B \cap T)$, and so $|X| = 3$. Then it follows $B \cap D = \emptyset$. Noticing that $D \cap S = \emptyset$, we have that $D = A \cap D = \{z\}$, which contradicts that $|D| \geq 2$. Therefore, $xz \in E_R(G)$. This completes the proof. $\square$

3.2 Removable Edges in a Cycle

Before we present and prove the main results of this chapter, we introduce the following definition.

Definition 3.2.1. *Let C be a cycle of a 4-connected graph G and H a sub-*

graph of G belonging to $\Re$. If C contains an inner vertex of H, then we say that C passes through H.

Now we present our main results.

Theorem 3.2.1. *Let G be a 4-connected graph and C a cycle of G. If C does not pass through any subgraph of G belonging to $\Re$, then there are at least two removable edges of G in C.*

Proof. By contradiction. Assume that C does not pass through any subgraph of G belonging to $\Re$, and there is at most one removable edge of G in C. Let $F = E(C) \cap E_R(G)$, so $|F| \leq 1$. Denote $E(C) - F$ by E_0. We consider the separating group $(uw, S'; A', B')$ such that $u \in A', w \in B'$ and $uw \in E_0$. Since $|F| \leq 1$ we know that $(E(A') \cup [A', S']) \cap F = \emptyset$ or $(E(B') \cup [S', B']) \cap F = \emptyset$. Without loss of generality, we may assume that $(E(A') \cup [A', S']) \cap F = \emptyset$. Since A' is an E_0-edge-vertex-cut fragment, A' must contain an E_0-edge-vertex-cut end-fragment as its subgraph, say A. Then we have that $(E(A) \cup [A, S]) \cap F = \emptyset$, and we consider separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$. We distinguish two main cases and a number of subcases.

Case 1. $|A| = 2$.

Then A is a 1-edge-vertex-cut atom or a 2-edge-vertex-cut atom, say $A = \{x, z\}$. Let $S = \{a, b, c\}$.

Subcase 1.1. A is a 2-edge-vertex-cut atom.

Since $xy \in E(C)$ and C is a cycle of G, we have that $\{xa, xb, xc, xz\} \cap E(C) \neq \emptyset$. From Lemma 3.1.2 we know that $\{xa, xb, xc, xz\} \subset E_R(G)$, which contradicts that $(E(A) \cup [A, S]) \cap F = \emptyset$.

Subcase 1.2. A is a 1-edge-vertex-cut atom.

By noticing that C is a cycle of G and $([E(A) \cup [A, S]) \cap F = \emptyset$, obviously

$\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$. From Corollary 3.1.1 we know that x is an inner vertex of one of the subgraphs of G belonging to $\Re$. Since $xy \in E(C)$, this contradicts the assumption that C does not pass through any subgraph of G belonging to $\Re$.

Case 2. $|A| \geq 3$.

We will distinguish the following subcases.

Subcase 2.1. There exists an $xz \in E_0 \cap E(A \cup [A, S])$.

Then obviously $z \notin S$; otherwise, we would have $|A| = 2$, a contradiction to $|A| \geq 3$. We take the separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1, z \in B_1$. Then we have that $x \in A \cap A_1, z \in A \cap B_1$. Let

$$X_1 = (A_1 \cap S) \cup (S \cap S_1) \cup (A \cap S_1)$$

$$X_2 = (A \cap S_1) \cup (S \cap S_1) \cup (B_1 \cap S)$$

$$X_3 = (B_1 \cap S) \cup (S \cap S_1) \cup (B \cap S_1)$$

$$X_4 = (B \cap S_1) \cup (S \cap S_1) \cup (A_1 \cap S)$$

If $y \in B \cap S_1$, from Theorem 2.1.2 we know that $|A_1| = 2$, say $A_1 = \{x, v_1\}$. We claim that A_1 is a 1-edge-vertex-cut atom; otherwise, A_1 is a 2-edge-vertex-cut atom, and then, by Lemma 3.1.2 we get $xy \in E_R(G)$, a contradiction. From Corollary 3.1.1 we know that x is an inner vertex of some subgraph of G belonging to $\Re$, a contradiction to the assumption. Therefore, $y \notin B \cap S_1$, and so $y \in A_1 \cap B$. Since $A \cap B_1 \neq \emptyset$, we have that X_2 is a vertex-cut of $G - xz$, and so $|X_2| \geq 3$. By analogous arguments, we can deduce that $|X_4| \geq 3$. Since $|X_2| + |X_4| = |S| + |S_1| = 6$, we get that $|X_2| = |X_4| = 3$, and so $|A_1 \cap S| = |A \cap S_1|, |B \cap S_1| = |B_1 \cap S|$. We claim that $A \cap B_1 = \{z\}$. Otherwise, $|A \cap B_1| \geq 2$. Then $(xz, X_2; A \cap B_1, A_1 \cup B)$ is a separating group of G and $xz \in E_0$. It is easy to see that $A \cap B_1$ is an E_0-edge-vertex-cut fragment contained in A, which contradicts that A is an E_0-edge-vertex-cut end-fragment of G. Therefore, $A \cap B_1 = \{z\}$. Since $|B_1| \geq 2$ and B_1 is a connected subgraph of G, we obtain $B_1 \cap S \neq \emptyset$.

Subcase 2.1.1. $|B_1 \cap S| = |B \cap S_1| = 3$.

Then $|X_1| = 0$, and so $\{z, y\}$ would be 2-vertex-cut of G, a contradiction.

Subcase 2.1.2. $|B_1 \cap S| = |B \cap S_1| = 2$.

Since X_1 is a vertex-cut of $G-xy-xz$, $|X_1| \geq 2$. Noticing that $|S| = |S_1| = 3$, we have that $|A \cap S_1| = |A_1 \cap S| = 1, S \cap S_1 = \emptyset$. We claim that $A \cap A_1 = \{x\}$. Otherwise, $|A \cap A_1| \geq 2$. Then $\{x\} \cup X_1$ is a 3-vertex-cut of G, a contradiction. Let $A \cap S_1 = \{a\}, A_1 \cap S = \{b\}, S \cap B_1 = \{v_1, v_2\}$. From $A \cap B_1 = \{z\}$ we deduce that $\Gamma_G(z) = \{x, a, v_1, v_2\}$. We claim that $ab \in E(G)$. Otherwise, $\{x, v_1, v_2\}$ is a 3-vertex-cut of G, a contradiction. We claim that $av_1, av_2 \in E(G)$. Otherwise, without loss of generality, we may assume that $av_1 \notin E(G)$. Let $A' = \{x, a\}, S' = \{b, z, v_2\}, B' = G - xy - S' - A'$. Then $(xy, S'; A', B')$ is a separating group of G. Since $xy \in E_0$, A' is an E_0-edge-vertex-cut fragment contained in A, which contradicts that A is an E_0-edge-vertex-cut end-fragment. So, $av_1, av_2 \in E(G)$, and hence $\Gamma_G(a) = \{x, z, b, v_1, v_2\}$. Let $S_0 = \{x, v_1, v_2\}, A_0 = \{a, z\}, B_0 = G - ab - S_0 - A_0$. Then $(ab, S_0; A_0, B_0)$ is a separating group of G, and so $ab \in E_N(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we consider the corresponding separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Since $axza, av_1za, av_2za$ are 3-cycles of G, we have that $x, v_1, v_2 \in S'$. Since $xz \in E_N(G)$, from Theorem 2.1.2 it follows that $|B'| = 2$, say $B' = \{z, u\}$. Then, $uzxu$ is a 3-cycle of G, which is impossible. Hence $az \in E_R(G)$.

Since $(E(A) \cup [A, S]) \cap F = \emptyset$ and C is a cycle of G, we get that $\{zv_1, zv_2\} \cap E_N(G) \neq \emptyset$. Without loss of generality, we may assume that $zv_1 \in E_N(G)$. We take the separating group $(zv_1, T; C', D')$ such that $z \in C', v_1 \in D'$. Then $z \in C' \cap B_1, v_1 \in B_1 \cap D'$. Obviously, $a \in S_1 \cap T$. Let

$$Y_1 = (A_1 \cap T) \cup (S_1 \cap T) \cup (C' \cap S_1)$$

$$Y_2 = (C' \cap S_1) \cup (S_1 \cap T) \cup (B_1 \cap T)$$

$$Y_3 = (B_1 \cap T) \cup (S_1 \cap T) \cup (S_1 \cap D')$$

$$Y_4 = (D' \cap S_1) \cup (S_1 \cap T) \cup (A_1 \cap T)$$

We distinguish the following cases to prove the Subcase 2.1.2.

(1.) $x \in A_1 \cap C'$. Then Y_1 is a vertex-cut of $G - xz$, and so $|Y_1| \geq 3$. By similar arguments, we have that $|Y_3| \geq 3$. Since $|Y_1| + |Y_3| = |S_1| + |T| = 6$, we conclude that $|Y_1| = |Y_3| = 3$ and $|A_1 \cap T| = |S_1 \cap D'|, |S_1 \cap C'| = |B_1 \cap T|$. Since $a \in S_1$, from Theorem 2.1.4 we know $b \notin T \cup S_1$. Since $bx, zv_2 \in E(G)$, we have that $b \in A_1 \cap C'$ and $v_2 \notin D' \cap B_1$. From $\Gamma_G(a) = \{v_1, v_2, z, x, b\}$, we know that $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$. Then we have that $|A_1 \cap T| = |S_1 \cap D'| = 0, 1$ or 2. Next we distinguish the following cases according to the value $|A_1 \cap T|$ and $|S_1 \cap D'|$.

(1.1.) $|A_1 \cap T| = |D' \cap S_1| = 2$. Then $|S_1 \cap C'| = |B_1 \cap T| = 0$. Since $zv_2 \in E(G)$, we have $v_2 \in B_1 \cap C'$, and hence $\{a, z\}$ is a 2-vertex-cut of G, a contradiction.

(1.2.) $|A_1 \cap T| = |D' \cap S_1| = 1$. Then $|S_1 \cap T| \leq 2$. First, we claim that $B_1 \cap D' = \{v_1\}$. Otherwise, $|B_1 \cap D'| \geq 2$. Then from $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$, we can conclude that $\{v_1\} \cup (Y_3 - \{a\})$ is a 3-vertex-cut of G, a contradiction. So, $B_1 \cap D' = \{v_1\}$. Let $D' \cap S_1 = \{u_1\}$. If $A_1 \cap D' \neq \emptyset$, from $\Gamma_G(a) = \{x, z, b, v_1, v_2\}$ we get that $A_1 \cap D' \cap \Gamma_G(a) = \emptyset$, and so $Y_4 - \{a\}$ is a vertex-cut of G with cardinality less than 4, a contradiction. Therefore, $A_1 \cap D' = \emptyset$. Then $au_1 \in E(G)$. However, it is easy to see that $u_1 \notin \{x, z, b, v_1, v_2\}$, a contradiction.

(1.3.) $|D' \cap S_1| = |A_1 \cap T| = 0$. Since D' is a connected subgraph of G, we have that $A_1 \cap D' = \emptyset$. Then $|D'| = |D' \cap B_1| \geq 2$. Since $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$, by noticing that $|Y_3| = 3$, we have that $\{v_1\} \cup (Y_3 - \{a\})$ is a 3-vertex-cut of G, a contradiction.

(2.) $x \in A_1 \cap T$. From Theorem 2.1.2 we have that $|C'| = 2$. Since C' is a connected subgraph of G, we have that $A_1 \cap C' = \emptyset$. If $S_1 \cap C' \neq \emptyset$, since $a \in S_1 \cap T$, we have $|D' \cap S_1| \leq 1$. Noticing that Y_3 is a vertex-cut of $G - zv_1$, we have that $|Y_3| \geq 3$, and so $|B_1 \cap T| = 1, A_1 \cap T = \{x\}$. Obviously, $|Y_4| = 3$, and hence $A_1 \cap D' = \emptyset$, and so $A_1 = \{x\}$, which contradicts $|A_1| \geq 2$. So $S_1 \cap C' = \emptyset$, and $|B_1 \cap C'| = 2$. Since $A_1 \cap T \neq \emptyset$, obviously, $\{z\} \cup (T - \{x\})$ is a vertex-cut with cardinality less than 4, a contradiction. This completes the proof of Subcase 2.1.2.

Subcase 2.1.3. $|B_1 \cap S| = |B \cap S_1| = 1$.

Then $|S \cap S_1| \leq 2$. We claim that $|S \cap S_1| < 2$. Otherwise, $|S \cap S'| = 2$, implying $A \cap S_1 = \emptyset = S \cap A_1$. If $|A \cap A'| \geq 2$, then $\{x\} \cup (S \cap S_1)$ is a vertex-cut of G with cardinality less than 4, a contradiction. So $A \cap A_1 = \{x\}$. Note that $|X_2| = 3$. If $|A \cap B_1| \geq 2$, then by arguments similar to that used in Subcase 2.1, $A \cap B_1$ is an E_0-edge-vertex-cut fragment contained in A, which contradicts that A is an E_0-edge-vertex-cut end-fragment. Hence, $A \cap B_1 = \{z\}$, and so $|A| = 2$, which contradicts $|A| \geq 3$. Therefore, $|S \cap S_1| \leq 1$, and we have $|X_3| \leq 3$. Since G is 4-connected, we have $B \cap B_1 = \emptyset$. Since $A \cap B_1 = \{z\}$, we have that $|B_1| = 2$ and B_1 is a 1-edge-vertex-cut atom of G, say $B_1 = \{z, u\}$. Since C is a cycle and $(E(A) \cup [A, S]) \neq \emptyset$, we have that z is incident with at least two unremovable edges. From Corollary 3.1.1 we know that z is an inner vertex of some subgraph of G belong to $\Re$, which contradicts the assumption that C does not pass through any subgraph of G belonging to $\Re$. This completes the proof of Theorem 3.2.1.$\square$

The following is our another main result of this chapter.

Theorem 3.2.2. *Let G be a 4-connected graph and C a cycle of G. If C passes through precisely one subgraph of G belonging to $\Re$, then there exists at* *least one removable edge of G in C.*

Proof. By contradiction. Assume that $E(C) \subset E_N(G)$. Let C pass through a subgraph H of G that belongs to $\Re$, (see Definitions 1.2.1 through 1.2.6). If H is a maximal l-belt, from the assumption it is easy to see that $\{x_2x_1, y_{l+1}y_{l+2}\} \cap E(C) \neq \emptyset$. If $x_2x_1 \in E(C)$, by letting $S = \{y_{l+2}, x_{l+2}, y_1\}, e = x_2x_1, B = \{x_2, \cdots, x_{l+1}, y_2, \cdots, y_{l+1}\}, A = G - e - S - B$, $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H; if $y_ly_{l+1} \in E(C)$, by letting $S = \{x_1, y_1, x_{l+2}\}, e = y_{l+1}y_{l+2}, B = \{x_2, \cdots, x_{l+1}, y_2, \cdots, y_{l+1}\}, A = G - e - S - B$, $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H. If H is a maximal l-co-belt, similarly, we have that $\{x_1x_2, y_1y_2\} \cap E(C) \neq \emptyset$. If $x_1x_2 \in E(C)$, by letting $S = \{y_{l+2}, x_{l+3}, y_1\}, e = x_2x_1, B = \{x_2, \cdots, x_{l+2}, y_2, \cdots, y_{l+1}\}, A = G - e - S - B$, $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H; if $y_1y_2 \in E(C)$, by letting $S = \{y_{l+2}, x_{l+3}, x_2\}, e = y_2y_1, B = \{x_3, \cdots, x_{l+2}, y_2, \cdots, y_{l+1}\}, A = G-e-S-B$, $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H. If H is a maximal l-bi-fan ($l \geq 1$), by letting $S = \{a, b, x_{l+3}\}, e = x_2x_1, B = \{x_2, \cdots, x_{l+2}\}, A = G - e - S - B$, $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H. If H is a helm, by letting $e = x_1v_1, S = \{v_2, v_3, v_4\}, B = \{a, x_1, x_2, x_3, x_4\}, A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H. If H is a W-framework, then C must pass through x_1x_2, x_2x_3. In this case, by letting $e = x_2x_1, S = \{x_3, y_4, y_2\}, B = \{x_2, y_3\}, A = G - e - S - B$, $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H. If H is a W'-framework, by noticing that $\{x_1x_2, x_2y_2\} \cap E(C) \neq \emptyset$, then if $x_1x_2 \in E(C)$, by letting $S = \{y_2, x_3, y_4\}, B = \{x_2, y_3\}, A = G - x_1x_2 - S - B$, $(x_1x_2, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H; if $x_2y_2 \in E(C)$, by letting $S = \{x_1, y_3, v\}$ such that $v \in \Gamma_G(x_3)$, $B = \{x_2, x_3\}, A = G - x_2y_2 - S - B$, $(x_2y_2, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H.

Let $E_0 = E(C)$. Then A is an E_0-edge-vertex-cut fragment of G such that it does not contain any inner vertex of H. Obviously, A contains an E_0-edge-vertex-cut end-fragment as its subgraph, say A'. It is easy to see that A' does not contain any inner vertex of H. To complete the proof of Theorem 3.2.2, by an argument analogous to that used in the proof of Theorem 3.2.1, we can show that A' contains an inner vertex of some subgraph of G belonging to $\Re$. $\square$

Finally, we construct examples, see figure 3.1, 3.2, 3.3, to show that the lower bounds in Theorems 3.2.1 and 3.2.2 are in some sense best possible. We also give an example to show that the conditions in Theorems 3.2.1 and 3.2.2 are not improved.

Let F be a maximal k-bi-fan such that $V(F) = \{a, b, z_1, z_2, \cdots, z_{k+3}\}$ and $E(F) = \{z_1z_2, z_2z_3, \cdots, z_{k+2}z_{k+3}, az_2, az_3, \cdots, az_{k+2}, bz_2, \cdots, bz_{k+2}\}$ where $k \geq 1$. Let L be a maximal l-belt such that $V(L) = \{x_1, x_2, \cdots, x_{l+2}, y_1, y_2, \cdots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$ where $E_1(H) = \{x_1x_2, x_2x_3, \cdots, x_{l+1}x_{l+2}, y_1y_2, y_2y_3, \cdots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \cdots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}\}$, in which $l \geq 1$.

Example 3.2.1 Identify vertex a with x_1, vertex b with y_{l+2}, vertex z_{k+3} with x_{l+2}, and vertex z_1 with y_1, respectively. Denote the resulting graph by G_1. Let $G = G_1 + ab + y_1x_{l+2}$. It is easy to see that G is a 4-connected graph. First, let $A = \{x_3, x_4, \cdots, x_{l+1}, y_2, y_3, \cdots, y_{l+1}\}, S = \{x_2, x_{l+2}, y_1\}, B = G - by_{l+1} - S - A$. Then $(by_{l+1}, S; A, B)$ is a separating group of G, and so $by_{l+1} \in E_N(G)$. Since $y_1x_{l+2} \in E([S])$, from Theorem 2.1.4 we have that $y_1x_{l+2} \in E_R(G)$. Obviously, $(x_{l+2}z_{k+2}, S_1)$ is a separating pair such that $S_1 = \{a, b, z_2\}$, and (z_2y_1, S_2) is a separating pair such that $S_2 = \{a, b, x_{l+2}\}$. It is easy to see that $z_iz_{i+1} \in E_N(G)$ where $i = 2, \cdots, k+1$. Consider the cycle $C_1 = y_1x_{l+2}z_{k+2}z_{k+1}z_k \cdots z_2y_1$. Then C_1 only passes through one subgraph of G belonging to $\Re$, and C_1 has only one removable edge y_1x_{l+2} of G. This shows that the result of Theorem 3.2.2 is in some sense best possible. See figure 3.1.

Example 3.2.2 First, delete the vertices z_1, z_{k+3} from F. Then, identify vertex z_2 with x_1, vertex z_{k+2} with y_{l+2}, respectively. Denote the resulting graph

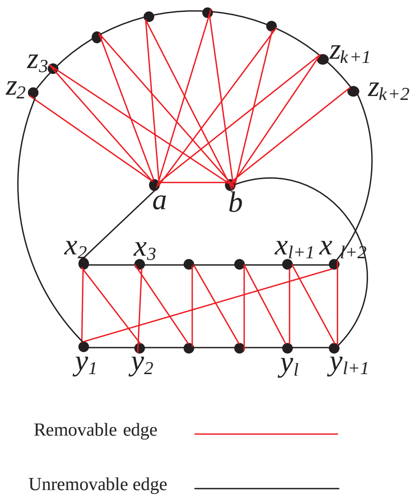

Figure 3.1:

by G_2. Let $G = G_2 + ab + ay_1 + bx_{l+2} + y_1x_{l+2}$. It is easy to see that G is a 4-connected graph. Let $A = \{x_3, \cdots, x_{l+1}, y_2, \cdots, y_{l+1}\}, S = \{y_1, x_{l+2}, x_2\}, B = G - z_{k+2}y_{l+1} - S - A$. Then $(z_{k+2}y_{l+1}, S; A, B)$ is a separating group of G, and so $z_{k+2}y_{l+1} \in E_N(G)$. Since $y_1x_{l+2} \in E([S])$, from Theorem 2.1.4 we have that $y_1x_{l+2} \in E_R(G)$. Obviously, (z_2x_2, S_1) is a separating group of G such that $S_1 = \{a, b, z_{k+2}\}$, and so $z_2x_2 \in E_N(G)$. By a similar argument, we get that $ay_1, bx_{l+2} \in E_N(G)$. Since $ab \in E([S_1])$, we have $ab \in E_R(G)$. Consider the cycle $C_2 = abx_{l+2}y_1a$. Then C_2 does not pass through any subgraph of G belonging to $\Re$, and C_2 has exactly two removable edges ab, y_1x_{l+2} of G. This shows that the result of Theorem 3.2.1 is in some sense best possible. See figure 3.2.

The following example shows that if a cycle C of 4-connected G passes through two subgraphs of G belonging to $\Re$, then it may not contain any removable edge of G.

Example 3.2.3 First, delete the vertex z_{k+3} from F. Then identify vertex a with x_1, vertex z_{k+2} with y_{l+2}, vertex z_1 with y_1, respectively. Join vertice b and x_{l+2}. Denote the resulting graph by G_3. Let $G = G_3 + ab + y_1x_{l+2}$. It is easy to see that G is a 4-connected graph. Consider the cycle $C_3 =$

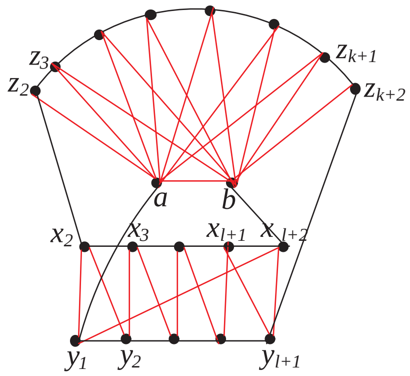

Figure 3.2:

$y_1y_2\cdots z_{l+2}z_{l+1}\cdots z_2y_1$. Then C_3 passes through two subgraphs of G belonging to $\Re$. It is easy to see that $E(C_3) \subset E_N(G)$, and so C_3 does not contain any removable edge of G. This in some sense shows that the conditions of Theorems 3.2.1 and 3.2.2 are best possible. See figure 3.3.

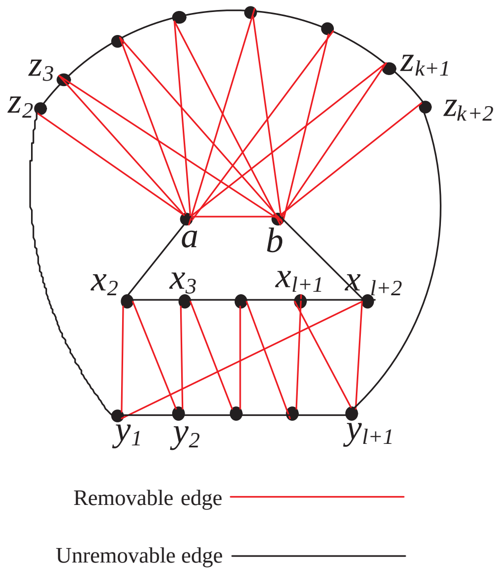

Figure 3.3:

Chapter 4

Removable Edges in a Longest Cycle of a 4-Connected Graph

In this chapter we obtain results on removable edges in a longest cycle of a 4-connected graph G. We also show that for a 4-connected graph G of minimum degree at least 5 or girth at least 4, any edge of G is removable or contractible.

4.1 Some Preliminary Results

Before giving our main results, we first show the following results.

Theorem 4.1.1. *Let G be a 4-connected graph with $|G| \geq 8$ such that $\delta(G) \geq 5$ or $g(G) \geq 4$. Then any edge of G is removable or contractible.*

Proof. By contradiction. Suppose there exists an edge e of G such that $e \in E_N(G)$ and $e \notin E_C(G)$. Then we will deduce contradictions as follows.

Let $e = xy$. Since $e \notin E_C(G)$, there exists a 4-vertex-cut T of G such that $e \in [T]$. Let $G - T = C \cup D$ such that C is the union of at least one but not of all the components of $G - T$ and $D = G - T - C$. Since $e \in E_N(G)$, by Theorem 2.1.2 there exists a separating group $(e, S; A, B)$ of G such that $x \in A, y \in B$. It is easy to see that $x \in A \cap T, y \in B \cap T$. Let

$$X_1 = (C \cap S) \cup (T \cap S) \cup (A \cap T),$$

$$X_2 = (A \cap T) \cup (T \cap S) \cup (S \cap D),$$

$$X_3 = (D \cap S) \cup (T \cap S) \cup (B \cap T),$$

$$X_4 = (T \cap B) \cup (T \cap S) \cup (S \cap C).$$

From $|G| \geq 8$ we know that there exists a vertex $v \in V(G)$ satisfying $v \notin S \cup T$. Symmetrically, we may assume $v \in A \cap C$. Since $A \cap C \neq \emptyset$ and G is 4-connected, we have that X_1 is a vertex-cut of G, and so $|X_1| \geq 4$. Since $|X_1| + |X_3| = |S| + |T| = 7$, we have that $|X_3| \leq 3$. Since G is 4-connected, we have that $B \cap D = \emptyset$. We distinguish the following cases to prove the theorem.

Case 1. $B \cap C \neq \emptyset$.

Then X_4 is a vertex-cut of G, and so $|X_4| \geq 4$. From $|X_2| + |X_4| = |S| + |T| = 7$, we get that $|X_2| \leq 3$. Since G is 4-connected, we have $A \cap D = \emptyset$. Thus, $D = D \cap S$, and so $D \cap S \neq \emptyset$. Noticing that $|S| = 3$, we have $|S \cap (C \cup T)| \leq 2$. Hence, $|X_1| \geq 4$ and $|X_4| \geq 4$. So, we have that $|A \cap T| \geq 2$ and $|B \cap T| \geq 2$ hold. Noticing that $|T| = 4$, we have that $|A \cap T| = |B \cap T| = 2$ and $|S \cap T| = 0$. Thus, $|S \cap C| = 2$, and so $|S \cap D| = 1$, i.e., $|D| = 1$. Let $D = \{u\}$. Then $xyux$ is a triangle of G, $d(u) = 4$ and $g(G) = 3$, a contradiction.

Case 2. $B \cap C = \emptyset$.

We have $B = B \cap T$. From $|X_1| \geq 4$, we get that $|S \cap C| \geq |B \cap T|$. Since $|B| \geq 2$, we have that $|B \cap T| \geq 2$. If $|B \cap T| \geq 3$, noticing that $|T| = 4$, then we have that $S \cap T = \emptyset$ and $A \cap T = \{x\}$. Based on the above arguments, we have $|S \cap C| \geq 3$. Noticing that $|S| = 3$, we have that $S \cap D = \emptyset$ and

$|X_2| = 1$, and so $A \cap D = \emptyset$. Consequently, $D = \emptyset$, a contradiction. Therefore, $|B \cap T| = 2$. Noticing that $|S| = 3$, we have $|S \cap (T \cup D)| \leq 1$. From $|B \cap T| = 2$, we get that $|A \cap T| \leq 2$, and so $|X_2| \leq 3$. Therefore, $A \cap D = \emptyset$. Then $D = D \cap S$, and so $D \cap S \neq \emptyset$. Noticing that $|S \cap C| \geq 2$ and $|S| = 3$, we have $|D \cap S| = 1$. Let $D \cap S = \{u\}$. Then $xyux$ is a triangle of G, $d(u) = 4$ and $g(G) = 3$, a contradiction. This completes the proof. □

From the proof of Theorem 4.1.1 we get the following result.

Corollary 4.1.1. *Let G be a 4-connected graph with $|G| \geq 8$. If there exists an edge $e \in E(G)$ such that $e \in E_N(G)$ and $e \notin E_C(G)$, then $\delta(G) = 4$ and e is on a triangle of G.* □

Before we prove our main result, we prove the following lemma.

Lemma 4.1.1. *Let G be a 4-connected graph with $|G| \geq 8$ and C be a longest cycle of G. Let $E(C) \subset E_N(G)$ and $E_0 = E(C)$. Suppose $x_1x_2 \in E(C)$ and $(x_1x_2, S; A, B)$ is a separating group such that $x_2 \in A, x_1 \in B$ and A is an E_0-edge-vertex-cut end-fragment. Then there are vertices $x, z, u, v \in V(C)$ such that $xz \in E(G)$(maybe $xz \notin E(C)$), $d(x) = d(z) = 4, \Gamma_G(x) \cap \Gamma_G(z) = \{u, v\}$ and $\{x, z\} \cap A \neq \emptyset$ and $\{x, z\} \cap B = \emptyset$.*

Proof. Since $E(C) = E_0$, the edge-vertex-cut fragment corresponding to any edge e on C is an E_0-edge-vertex-cut fragment. Take any edge x_1x_2 on C. Since $x_1x_2 \in E_N(G)$, by Theorem 2.1.1 there is a separating group $(x_1x_2, S; A, B)$ such that $x_2 \in A$ and $x_1 \in B$. It is easy to see that every E_0-edge-vertex-cut fragment contains such an end-fragment as a subset. Without loss of generality, from the arbitrariness of e on C we can assume that A is such an end-fragment. Since C is a cycle, we have that $(E(A) \cup [A, S]) \cap E(C) \neq \emptyset$. Take an edge x_2x_3 in the intersection. Since $x_2x_3 \in E(C) \subset E_N(G)$, we can consider a separating group $(x_2x_3, S'; A', B')$ such that $x_2 \in A'$ and $x_3 \in B'$. Note that $x_2 \in A \cap A'$. Let

$$X_1 = (A' \cap S) \cup (S' \cap S) \cup (A \cap S'),$$

$$X_2 = (A \cap S') \cup (S' \cap S) \cup (S \cap B'),$$

$$X_3 = (B' \cap S) \cup (S' \cap S) \cup (B \cap S'),$$

$$X_4 = (S' \cap B) \cup (S' \cap S) \cup (S \cap A').$$

We distinguish the following cases to complete the proof.

Case 1. $x_3 \in A \cap B'$ and $x_1 \in A' \cap B$.

Since $A' \cap B \neq \emptyset$, we know that X_4 is a vertex-cut of the graph $G - x_1x_2$. Since G is 4-connected and so $G - x_1x_2$ is 3-connected, we have that $|X_4| \geq 3$. Similarly, X_2 is a vertex-cut of $G - x_2x_3$ and so $|X_2| \geq 3$. Since $|X_2| + |X_4| = |S| + |S'| = 6$, we get that $|X_2| = |X_4| = 3$, and hence $|A' \cap S| = |A \cap S'|$ and $|B \cap S'| = |B' \cap S|$. Since $|S| = |S'| = 3$, we can distinguish the following four subcases for the value $|B \cap S'| = |B' \cap S|$.

Subcase 1.1. $|B \cap S'| = |B' \cap S| = 3$.

Note that $|S| = |S'| = 3$. This implies that $|X_1| = 0$. Hence, $\{x_1, x_3\}$ is a 2-vertex-cut of G, which contradicts that G is 4-connected.

Subcase 1.2. $|B \cap S'| = |B' \cap S| = 2$.

We claim that $S \cap S' = \emptyset$. If not, since $|S| = |S'| = 3$, we get that $|S \cap S'| = 1$. Then $A' \cap S = A \cap S' = \emptyset$, and hence $|X_1| = 1$. Then $X_1 \cup \{x_1, x_3\}$ is a 3-vertex-cut of G, which contradicts that G is 4-connected. Therefore, $S \cap S' = \emptyset$, and so $|A' \cap S| = |A \cap S'| = 1$. This implies that $|X_1| = 2$. We claim that $A \cap A' = \{x_2\}$. Otherwise, $|A \cap A'| \geq 2$. Then since $|X_1| = 2$, it is easy to see that $\{x_2\} \cup X_1$ is a 3-vertex-cut of G, a contradiction. Therefore, $A \cap A' = \{x_2\}$. Next, we claim that $A \cap B' = \{x_3\}$. Otherwise, $|A \cap B'| \geq 2$. Let $A_1 = A \cap B'$, $S_1 = X_2$ and $B_1 = G - x_2x_3 - S_1 - A_1$. Then $(x_2x_3, S_1; A_1, B_1)$ is a separating group of G. Since $x_2x_3 \in E_0$, A_1 is an E_0-edge-vertex-cut fragment of G. Since $A_1 \subset A$, this contradicts that A is an E_0-edge-vertex end-fragment. Therefore, $A \cap B' = \{x_3\}$. Let $A \cap S' = \{a\}$, $A' \cap S = \{b\}$ and $B' \cap S = \{u, v\}$. We claim $ab \in E(G)$. If not, then $\{u, v, x_2\}$ would be a 3-vertex-cut of G, which contradicts that G is 4-connected. Therefore, $ab \in E(G)$. It is easy to see that $\Gamma_G(x_2) = \{x_1, x_3, a, b\}$ and $\Gamma_G(x_3) = \{a, u, v, x_2\}$. First, we let $e_1 = ab$, $S_1 = \{x_2\} \cup (B \cap S')$, $A_1 = A' \cap (B \cup S)$ and $B_1 = G - e_1 - S_1 - A_1$. Then $(e_1, S_1; A_1, B_1)$ is a separating group of G, and so $ab \in E_N(G)$. Next, we claim $ax_3 \in E_R(G)$. If not, $ax_3 \in E_N(G)$, and hence there is a corresponding separating group $(ax_3, S_1; A_1, B_1)$ such that $a \in A_1$ and $x_3 \in B_1$. Since ax_2x_3a is a triangle of G, we have that $x_2 \in S_1$. Since $x_2x_3 \in E_N(G)$, by Theorem 2.1.2 we have that $|B_1| = 2$, say $B_1 = \{v_1, x_3\}$. Then it is easy to see that $v_1x_2x_3v_1$ is a triangle of G and $v_1 \neq a$, which is impossible in G. Therefore, $ax_3 \in E_R(G)$. Since C is a cycle, $x_3 \in V(C)$ and $E(C) \subset E_N(G)$, we have that $\{x_3u, x_3v\} \cap E_N(G) \neq \emptyset$. Without loss of generality, we assume that $x_3u \in E_N(G)$.

We claim that $au \notin E(G)$. By contradiction, suppose $au \in E(G)$. Since $x_3u \in E_N(G)$, there is a corresponding separating group $(x_3u, T_1; C_1, D_1)$ such that $x_3 \in C_1$ and $u \in D_1$. So, $x_3 \in C_1 \cap B'$ and $u \in B' \cap D_1$. Since ax_3ua is a triangle of G, we have $a \in T_1$, and so $a \in S' \cap T_1$. Let

$$Y_1 = (A' \cap T_1) \cup (S' \cap T_1) \cup (C_1 \cap S'),$$

$$Y_2 = (C_1 \cap S') \cup (S' \cap T_1) \cup (B' \cap T_1),$$

$$Y_3 = (B' \cap T_1) \cup (S' \cap T_1) \cup (S' \cap D_1),$$

$$Y_4 = (D_1 \cap S') \cup (S' \cap T_1) \cup (A' \cap T_1).$$

We distinguish the following cases to prove the claim.

(1.) $x_2 \in A' \cap C_1$. Then Y_1 is a vertex-cut of $G - x_2x_3$. Since G is 4-connected, we have $|Y_1| \geq 3$. Similarly, we have $|Y_3| \geq 3$. Since $|Y_1| + |Y_3| = |S'| + |T_1| = 6$, we have $|Y_1| = |Y_3| = 3$. Then $|A' \cap T_1| = |S' \cap D_1|$ and $|S' \cap C_1| = |B' \cap T_1|$ hold. Since $a \in S' \cap T_1$ and $ab \in E_N(G)$, by Theorem 2.1.4 we have that $b \notin T_1 \cup S'$. Since $bx_2 \in E(G)$, we have $b \in A' \cap C_1$. From $x_3v \in E(G)$ we know that $v \notin D_1 \cap B'$, and so $v \in B' \cap (C_1 \cup T_1)$. Clearly, $|A' \cap T_1| = |S' \cap D_1| \leq 2$. We discuss the following cases for the value $|A' \cap T_1| = |S' \cap D_1|$.

(1.1.) $|A' \cap T_1| = |S' \cap D_1| = 2$. Noticing that $|T_1| = |S'| = 3$ and $a \in S' \cap T_1$ we have that $|S' \cap C_1| = |B' \cap T_1| = 0$. Since avx_3a is a triangle of G, we have $v \in B' \cap C_1$, and so $|B' \cap C_1| \geq 2$. Then $\{a, x_3\}$ is a 2-vertex-cut of G, a contradiction.

(1.2.) $|A' \cap T_1| = |S' \cap D_1| = 1$. Then $|S' \cap T_1| \leq 2$. First, we claim that $B' \cap D_1 = \{u\}$. If not, then $|B' \cap D_1| \geq 2$. Since $\Gamma_G(a) = \{x_2, x_3, u, v, b\}$, by the foregoing arguments we have that $\Gamma_G(a) \cap (B' \cap D_1) = \{u\}$. So $\{u\} \cup (Y_3 - \{a\})$ is a 3-vertex-cut of G, a contradiction. Therefore, $B' \cap D_1 = \{u\}$. Let $D_1 \cap S' = \{u_1\}$. If $|S' \cap T_1| = 1$, i.e., $S' \cap T_1 = \{a\}$, then $|Y_4| = 3$. Since G is 4-connected, we have that $D_1 \cap A' = \emptyset$. So, $u_1 \in \Gamma_G(a)$. However, it is easy to see that $u_1 \notin \{x_2, x_3, b, u, v\}$, a contradiction. Therefore, $|S' \cap T_1| = 2$ must hold. Then $A' \cap D_1 = \emptyset$. If not, then $Y_4 - \{a\}$ is a 3-vertex-cut of G, a contradiction. So, $A' \cap D_1 = \emptyset$ and it is easy to see that $au_1 \in E(G)$. However, this would imply that $u_1 \in \{b, u, v, x_2, x_3\}$, a contradiction.

(1.3.) $|A' \cap T_1| = |S' \cap D_1| = 0$. Since D_1 is a connected subgraph of G, we have that $A' \cap D_1 = \emptyset$. From $|D_1| \geq 2$ we have that $|D_1 \cap B'| \geq 2$. Since $|Y_3| = |T_1| = 3$, by analogous arguments we have that $\Gamma_G(a) \cap (D_1 \cap B') = \{u\}$.

So $\{u\} \cup (Y_3 - \{a\})$ is a 3-vertex-cut of G, a contradiction.

(2.) $x_2 \in A' \cap T_1$. Since $x_3x_2 \in E_N(G)$, by Theorem 2.1.2 we have that $|C_1| = 2$. Since C_1 is a connected subgraph of G, we have that $A' \cap C_1 = \emptyset$. If $S' \cap C_1 \neq \emptyset$, then due to $|C_1| = 2$, we have that $|S' \cap C_1| = 1$. From $a \in S' \cap T_1$ we have that $|D_1 \cap S'| \leq 1$. Since Y_3 is a vertex-cut of $G - x_3u$, we have that $|Y_3| \geq 3$, and so $|B' \cap T_1| \geq 1$. Noticing that $|T_1| = 3$, we get that $A' \cap T_1 = \{x_2\}$ and $|Y_4| = 3$. Since G is 4-connected, we have that $A' \cap D_1 = \emptyset$. Hence, $A' = \{x_2\}$, which contradicts that $|A'| \geq 2$. If $S' \cap C_1 = \emptyset$, then $|B' \cap C_1| = 2$. Since $A' \cap T_1 \neq \emptyset$, we have that $|Y_2| = |T_1 \cap (B' \cup S')| \leq 2$, and so $\{x_3\} \cup Y_2$ is a vertex-cut of G with cardinality less than 4, a contradiction.

So we have proved and claim that $au \notin E(G)$. Let $A_1 = \{a, x_2\}$, $S_1 = \{x_3\} \cup (S - \{u\})$ and $B_1 = G - x_1x_2 - S_1 - A_1$. Then $(x_1x_2, S_1; A_1, B_1)$ is a separating group of G and $x_1x_2 \in E_0$. So, A_1 is an E_0-edge-vertex-cut fragment and $A_1 \subset A$, which contradicts that A is an E_0-edge-vertex-cut end-fragment. This complete the proof of Subcase 1.2.

Subcase 1.3. $|B \cap S'| = |B' \cap S| = 1$.

Then $|S \cap S'| \leq 2$. We distinguish the following cases for the value $|S \cap S'|$.

Subcase 1.3.1. $|S \cap S'| = 2$.

Then $|A' \cap S| = |A \cap S'| = 0$ and so $|X_1| = 2$. We claim that $A \cap A' = \{x_2\}$. If not, then $|A \cap A'| \geq 2$, and so $X_1 \cup \{x_2\}$ is a 3-vertex-cut of G, a contradiction. Hence, $A \cap A' = \{x_2\}$. Since $|X_2| = 3$, we claim that $A \cap B' = \{x_3\}$. Otherwise, $|A \cap B'| \geq 2$. Let $A_1 = A \cap B'$, $S_1 = X_2$ and $B_1 = G - x_2x_3 - S_1 - A_1$. Then $(x_2x_3, S_1; A_1, B_1)$ is a separating group of G. Since $x_2x_3 \in E_0$, A_1 is an E_0-edge-vertex-cut fragment and $A_1 \subset A$, which contradicts to that A is an E_0-edge-vertex-cut end-fragment. Hence, $A \cap B' = \{x_3\}$. Then we have that $d(x_2) = d(x_3) = 4$. Let $S \cap S' = \{a, b\}$. Obviously, we have that $ax_2, ax_3, bx_2, bx_3 \in E(G)$. Since C is a longest cycle of G, it is easy to see that

$a, b \in V(C)$. So in this case the conclusion of the lemma holds.

Subcase 1.3.2. $|S \cap S'| = 1$.

Then $|A' \cap S| = |A \cap S'| = 1$. Since $|X_2| = 3$, by an argument analogous to that used in Subcase 1.3.1 we can show that $A \cap B' = \{x_3\}$. Let $A \cap S' = \{a\}$, $S \cap S' = \{b\}$, $B' \cap S = \{c\}$ and $S' \cap B = \{u\}$. Since $|X_3| = 3$ and G is 4-connected, we have that $B \cap B' = \emptyset$. Obviously, we have that $d(x_3) = d(c) = 4, x_3c \in E(G)$, $\Gamma_G(x_3) \cap \Gamma_G(c) = \{a, b\}$ and $\{x_3, c\} \cap A \neq \emptyset$. If $x_3c \in E(C)$, since ax_3ca and bx_3cb are triangles, we have $a, b \in V(C)$, and so the conclusions of Lemma 4.1.1 hold. Hence, we may assume $x_3c \notin E(C)$. Then we have that $\{ax_3, bx_3\} \cap E(C) \neq \emptyset$, and so $c \in V(C)$. It suffices to prove $a, b \in V(C)$.

First we claim $a \in V(C)$. If not, then $ax_3, ac \notin E(C)$. Since $cx_3 \notin E(C)$ and $bx_3 \in E(C)$, we have $c \in V(C)$, and hence $bc, cu \in E(C)$. Noticing that $E(C) \subset E_N(G)$, we have $bc \in E_N(G)$. However, $bc \in E([S])$, and use Theorem 2.1.4 we have $bc \in E_R(G)$, a contradiction. Therefore, $a \in V(C)$.

Next we claim $b \in V(C)$. If not, then $bx_3, bc \notin E(C)$. Since $c \in V(C)$, we have that $ac, cu, ax_3 \in E(C)$. Consider a separating group $(ax_3, T'; C', D')$ such that $a \in C'$ and $x_3 \in D'$. Then $c \in T'$. Since $\Gamma_G(c) = \{x_3, a, b, u\}$ and $au \notin E(G)$, we have that $b \in C'$ and $u \in D'$. Noticing $ac \in E_N(G)$, using Theorem 2.1.2 we have that $C' = \{a, b\}$, and so $ab \in E(G)$. Since C is a longest cycle of G, we have $b \in V(C)$. So in this case the conclusions of Lemma 4.1.1 hold.

Subcase 1.3.3. $S \cap S' = \emptyset$.

Then we have that $|A \cap S'| = |A' \cap S| = 2$. Let $A \cap S' = \{a, b\}$, $B' \cap S = \{c\}$ and $B \cap S' = \{u\}$. Since $|X_3| = 2$, we have that $B \cap B' = \emptyset$. By an argument analogous to that used in Subcase 1.3.1 we can show that $A \cap B' = \{x_3\}$. Then $\Gamma_G(x_3) = \{x_2, a, b, c\}$ and $\Gamma_G(c) = \{x_3, a, b, u\}$. If $cx_3 \in E(C)$, by an argument analogous to that used in Subcase 1.3.2 we can deduce the conclusions of Lemma 4.1.1. If $cx_3 \notin E(C)$, we may assume that $ax_3 \in E(C)$. Then $c \in V(C)$. We claim $b \in V(C)$. If not, then $ac, uc \in E(C)$. Consider a separating group $(ax_3, T_1; C_1, D_1)$ such that $x_3 \in C_1$ and $a \in D_1$. Then $c \in T_1$. Since $ac \in E_N(G)$, using Theorem 2.1.2 we have that $|D_1| = 2$. Since $\Gamma_G(c) = \{x_3, a, b, u\}$, we have that $b \in D_1$ or $u \in D_1$. If $u \in D_1$, then $au \in E(G)$, contradicting that $a \in A$ and $u \in B$; if $b \in D_1$, then $ab \in E(G)$, and so $b \in V(C)$, a contradiction. Hence, $b \in V(C)$. Since $d(x_3) = d(c) = 4$ and $\Gamma_G(x_3) \cap \Gamma_G(c) = \{a, b\}$, in this case Lemma 4.1.1 is true.

Subcase 1.4. $B \cap S' = \emptyset = B' \cap S$.

Then $B \cap B' = \emptyset$, B' is an E_0-edge-vertex-cut fragment and $B' \subset A$, which contradicts that A is an E_0-edge-vertex-cut end-fragment. So, Subcase 1.4 does not occur.

Case 2. $x_3 \in B' \cap S$ and $x_1 \in A' \cap B$.

By arguments analogous to that used in Subcase 1.3.2 we can deduce that $A \cap A' = \{x_2\}, A \cap S' = \{a\}, S \cap A' = \{b\}, S \cap B' = \{x_3, u\}$ and $S \cap S' = \emptyset$. Hence $d(x_2) = d(a) = 4, ax_2 \in E(G)$ and $\Gamma_G(x_2) \cap \Gamma_G(a) = \{b, x_3\}$. Since $x_2x_3 \in E(C)$ and ax_2x_3a is a triangle, we have $a \in V(C)$. We claim $b \in V(C)$. If not, then $bx_2, ba \notin E(C)$, and so $ax_3, au \in E(C)$. Consider a separating group $(ax_3, T'; C', D')$ such that $a \in C'$ and $x_3 \in D'$, then $x_2 \in T'$. Since $\Gamma_G(x_2) = \{x_3, a, b, x_1\}$ and $ab \in E(G)$, we have that $b \in C'$ and $x_1 \in D'$. Noticing $x_2x_3 \in E_N(G)$, from Theorem 2.1.2 we have that $C' = \{x_3, x_1\}$, and so $x_1x_3 \in E(G)$, contradicting to that $x_1 \in A'$ and $x_3 \in B'$. Therefore, $b \in V(C)$, and so in this case lemma holds.

Case 3. $x_3 \in A \cap B'$ and $x_1 \in B \cap S'$.

By arguments analogous to that used in Case 2 we can deduce that $A \cap A' = \{x_2\}, S \cap A' = \{b\}, A \cap S' = \{a\}$ and $S \cap S' = \emptyset$. Since A is an E_0-edge-vertex-cut end-fragment, by arguments analogous to that used in Subcase 1.3.1 we can deduce that $A \cap B' = \{x_3\}$. Hence $d(x_2) = d(b) = 4$ and

$\Gamma_G(x_2) \cap \Gamma_G(b) = \{a, x_1\}$. Since $x_1x_2, x_2x_3 \in E(C)$ and bx_1x_2b and ax_2x_3a are triangles, we have $a, b \in V(C)$, and so in this case lemma 4.1.1 holds.

Case 4. $x_3 \in B' \cap S$ and $x_1 \in B \cap S'$.

By arguments analogous to that used in Case 2 we can deduce that $A \cap A' = \{x_2\}$, $S \cap A' = \{b\}$, $A \cap S' = \{a\}$ and $S \cap S' = \emptyset$. Hence, $d(x_2) = d(b) = 4$ and $\Gamma_G(x_2) \cap \Gamma_G(b) = \{a, x_1\}$. Since $x_1x_2, x_2x_3 \in E(C)$, and bx_1x_2b and ax_2x_3a are triangles, we have $a, b \in V(C)$. So in this case Lemma 4.1.1 holds. This completes the proof of the Lemma 4.1.1. $\square$

4.2 Removable Edges on Longest Cycles

Before proceeding, we introduce the following notations.

Definition 4.2.1. Let G be a 4-connected graph and H be a subgraph of G. If $V(H) = \{u, v, x, z\}$, $E(H) = \{xz, ux, vx, uz, vz\}$ and $d(x) = d(z) = 4$, then H is called a *bi-triangle*, and x, z are called its *inner vertices*. If a cycle C of G contains the vertices u, v, x and z, we say that C *passes through* the bi-triangle H.

We now have all the ingredients to present and prove the two main results of this chapter.

Theorem 4.2.1. *Let G be a 4-connected graph with $|G| \geq 8$. If a longest cycle C of G does not pass through any bi-triangle, then C contains at least two removable edges.*

Proof. By contradiction. Suppose C contains at most one removable edge of G. Let $F = E(C) \cap E_R(G)$. Then $|F| \leq 1$. Let $E_0 = E(C) - F$, and so $E_0 \neq \emptyset$. Then for an edge uw in E_0, there is a separating group $(uw, S'; A', B')$ of G such that $u \in A'$ and $w \in B'$. Since $|F| \leq 1$, we have that $(E(A') \cup [A', S']) \cap F = \emptyset$, or $(E(B') \cup [B', S']) \cap F = \emptyset$. Without loss of generality, we assume that $(E(A') \cup [A', S']) \cap F = \emptyset$. Since A' is an E_0-edge-vertex-cut fragment, there must exist an E_0-edge-vertex-cut end-fragment contained in A', say A. Then corresponding to A there is a separating group $(xy, S; A, B)$ of G such that $x \in A$, $y \in B$, $|S| = 3$ and $xy \in E_0$. Obviously, $(E(A) \cup [A, S]) \cap F = \emptyset$. Since C is a cycle of G, there exists an edge $xz \in E(C) \cap (E(A) \cup [A, S]) \neq \emptyset$. Analogously, we consider the separating group $(xz, S'; A', B')$ of G such that $x \in A', z \in B'$. By analogous arguments as used in the proof of Lemma 4.1.1 we can show that C passes through at least one bi-triangle, which contradicts the assumption of the theorem. This completes the proof. $\square$

Theorem 4.2.2. *Let G be a 4-connected graph with $|G| \geq 8$. If a longest cycle C of G passes through at most one bi-triangle, then C contains at least one removable edge.*

Proof. By contradiction. Suppose C does not contain any removable edge. Let $E_0 = E(C)$. If C does not pass through any bi-triangle, then by Theorem 4.2.1 the theorem holds. So, next we assume that C passes through precisely one bi-triangle H as defined in Definition 4.2.1. We consider an E_0-edge-vertex-cut end-fragment A and its corresponding separating group $(ww', S; A, B)$ of G such that $w \in A$ and $w' \in B$. If $xz = ww'$, we may assume that $x = w, z = w'$. Then from Lemma 4.1.1 there exists an inner vertex $x' \in A$ and $x' \in V(C)$. Let H' be the bi-triangle containing x' as its inner vertex. By Lemma 4.1.1 we know that C passes through H'. It is easy to see that $V(H') \subset V(A) \cup V(S)$, and so $z \notin V(H')$ and $H \neq H'$, which contradicts that C passes through only one bi-triangle. If $xz \neq ww'$, using Lemma 4.1.3 we have that $V(H) \subset V(A) \cup V(S)$, and so $V(H) \cap V(B) = \emptyset$. Then B must contain an E_0-edge-vertex-cut end-fragment B' satisfying $V(B') \cap V(H) = \emptyset$. Since B' is an E_0-edge-vertex-cut end-fragment, by Lemma 4.1.1 there are vertices $x', z', u', v' \in V(C)$ such that $x'z' \in E(G)$, $d(x') = d(z') = 4, \Gamma_G(x') \cap \Gamma_G(z') = \{u', v'\}$ and $\{x', z'\} \cap B' \neq \emptyset$. So C passes through the bi-triangle H' and $H' \neq H$, which contradicts that C passes through only one bi-triangle. This completes the proof. $\square$

Chapter 5

Removable Edges on a Hamilton Cycle in a 4-Connected Graph

In this chapter we study the distribution of removable edges on a Hamilton cycle in a 4-connected graph, and give examples to show that some results are in some sense best possible.

5.1 Some Preliminary Results

Before we give the main results of this chapter, we first show the following Lemma.

Lemma 5.1.1. *Let G be a 4-connected graph, $E_0 \subset E_N(G)$ and $E_0 \neq \emptyset$. Let $(xy, S; A, B)$ be a separating group of G such that $x \in A, y \in B, S = \{a, b, c\}, xy \in E_0$. If A is an E_0-edge-vertex end-fragment of G, and $|A| \geq 3$, then one of the following conclusions (i),(ii) or (iii) holds:*

(i) $(E(A) \cup [A, S]) \cap E_0 = \emptyset$.

(ii) *There exists a separating group $(x'y', S'; A', B')$ of G such that $x' \in A', y' \in B', x'y' \in E_0$, B' is a 1-edge-vertex-cut atom, and $|A \cap B'| = |B' \cap S| = 1$.*

(iii) *There exists a separating group $(xy', S'; A', B')$ of G such that $x \in A', y' \in B', xy' \in E_0$, $A \cap A' = \{x\}, |A \cap S'| = 1, A \cap B' = \{y'\}, |B' \cap S| = 2$.*

Proof. $(E(A) \cup [A, S]) \cap E_0 = \emptyset$. Then conclusion (i) holds. So next we may assume that $(E(A) \cup [A, S]) \cap E_0 \neq \emptyset$, and so we have that either $E(A) \cap E_0 \neq \emptyset$

or $[A, S] \cap E_0 \neq \emptyset$ holds. We distinguish the following cases to complete the proof.

Case 1. There exists an edge $uz \in E(A) \cap E_0$.

We consider the separating group $(uz, T; C, D)$ such that $u \in C, z \in D$. Then we have $u \in A \cap C, z \in A \cap D$. Let

$$X_1 = (C \cap S) \cup (S \cap T) \cup (A \cap T)$$

$$X_2 = (A \cap T) \cup (S \cap T) \cup (D \cap S)$$

$$X_3 = (D \cap S) \cup (S \cap T) \cup (B \cap T)$$

$$X_4 = (B \cap T) \cup (S \cap T) \cup (C \cap S)$$

We distinguish the following subcases to complete the proof of Case 1.

Subcase 1.1. $x \neq u$. Then we have $x \in A \cap C$, $A \cap T$, or $A \cap D$.

(1.) $x \in A \cap C$. Then we have that $y \in B \cap C$ or $B \cap T$.

(1.1.) $y \in B \cap C$. Since $A \cap D \neq \emptyset$, X_2 is a vertex-cut of $G - uz$. Since G is 4-connected, $|X_2| \geq 3$. By similar arguments, we get that $|X_4| \geq 3$. Noticing that $|X_2| + |X_4| = |S| + |T| = 6$, we have $|X_2| = |X_4| = 3$, and so $|S \cap C| = |A \cap T|, |B \cap T| = |D \cap S|$. First, we claim that $A \cap D = \{z\}$. Otherwise, $|A \cap D| \geq 2$. Let $A_1 = A \cap D, S_1 = X_2, B_1 = G - uz - S_1 - A_1$. Then $(uz, S_1; A_1, B_1)$ is a separating group of G. Since $uz \in E_0$, A_1 is an E_0-edge-vertex-cut fragment contained in A, which contradicts the fact that A is an E_0-edge-vertex-cut end-fragment. Hence $A \cap D = \{z\}$. Since $|D| \geq 2$ and D is a connected subgraph of G, $D \cap S \neq \emptyset \neq B \cap T$. Combining with $|D \cap S| = |B \cap T|$, next we make observations on $|D \cap S|$ as follows:

(1.1.1.) $|D \cap S| = |B \cap T| = 3$. Noticing $|S| = |T| = 3$, it is easy to see that

$|X_1| = 0$. Then $\{z, y\}$ is a 2-vertex-cut of G, a contradiction.

(1.1.2.) $|D \cap S| = |B \cap T| = 2$. Since X_1 is a vertex-cut of $G - uz - xy$ and G is 4-connected, we have $|X_1| \geq 2$, which implies $|S \cap C| = |A \cap T| = 1, |S \cap T| = 0$. Noticing $x, u \in A \cap C$, we have $|A \cap C| \geq 2$. Let $A_1 = A \cap C, S_1 = \{z\} \cup X_1$, $B_1 = G - xy - X_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G. Since $xy \in E_0$, A_1 is an E_0-edge-vertex-cut fragment contained in A, which contradicts the fact that A is an E_0-edge-vertex-cut end-fragment.

(1.1.3.) $|B \cap T| = |D \cap S| = 1$. Obviously, $|S \cap T| \leq 2$. We claim that $S \cap T \neq 2$. Otherwise, $|S \cap T| = 2$, and then $|C \cap S| = |A \cap T| = 0$. Let $A_1 = A \cap C, S_1 = (S \cap T) \cup \{z\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G. Since $xy \in E_0$, A_1 is an E_0-edge-vertex-cut fragment contained in A, which contradicts the fact that A is an E_0-edge-vertex-cut end-fragment. Hence $|S \cap T| \neq 2$, i.e., $|S \cap T| \leq 1$. Then we have $|X_3| \leq 3$, and so $B \cap D = \emptyset$. It is easy to see that D is a 1-edge-vertex-cut atom, and $|A \cap D| = 1, |S \cap D| = 1$, $|B \cap T| = 1$. Let $D = B', T = S', C = A', u = x', z = y'$. Then conclusion (ii) holds.

(1.2.) $y \in B \cap T$. Since $A \cap D \neq \emptyset$, X_2 is a vertex-cut of $G - uz$. So $|X_2| \geq 3$, and hence $|D \cap S| \geq |B \cap T| \geq 1$. Using $|S| = 3$, we have $|C \cap S| \leq 2$. Noticing that $|X_2| + |X_4| = |S| + |T| = 6$, it follows $|X_4| \leq 3$. Since G is 4-connected, we have $B \cap C = \emptyset$. If $C \cap S = \emptyset$, then $C = A \cap C$. It is easy to see that C is an E_0-edge-vertex-cut fragment contained in A, which contradicts the fact that A is an E_0-edge-vertex-cut end-fragment. Hence $C \cap S \neq \emptyset$. If $S \cap T \neq \emptyset$, then $|S \cap T| = 1$, and $|C \cap S| = |D \cap S| = 1$. Since $|D \cap S| \geq |B \cap T|$, we have $B \cap T = \{y\}$. Obviously, now we obtain $|X_3| = 3$, and so $B \cap D = \emptyset$. Hence $B = B \cap T = \{y\}$, which contradicts $|B| \geq 2$, and so $S \cap T = \emptyset$. If $|C \cap S| = 2$, then $|D \cap S| = 1$, and so $|B \cap T| = 1$. Now we have $|X_3| = 2$, so $B \cap D = \emptyset$, and hence $B = \{y\}$, which contradicts $|B| \geq 2$. Hence, $|C \cap S| = 1$, and so $|S \cap D| = 2$. If $|B \cap T| = 1$, by similar arguments, we get that $|B| = 1$, a contradiction. Hence, $|B \cap T| = 2$, then $|A \cap T| = 1$, and so $|X_1| = 2$. Noticing that $|A \cap C| \geq 2$, we let $A_1 = A \cap C, S_1 = X_1 \cup \{z\}, B_1 = G - xy - S_1 - A_1$, then $(xy, S_1; A_1, B_1)$ is a separating group of G. Since $xy \in E_0$, A_1 is an E_0-edge-vertex-cut fragment contained in A, which contradicts the assumption that A is an E_0-edge-vertex-cut end-fragment. So (1.2) does not occur.

(2.) $x \in A \cap T$. By Theorem 2.1.4, we know that $y \notin B \cap T$. By symmetry, we may assume that $y \in B \cap C$. Since $A \cap D \neq \emptyset$, X_2 is a vertex-cut of $G - uz$, and so $|X_2| \geq 3$. By similar arguments, we get that $|X_4| \geq 3$. Since $|X_2| + |X_4| = |S| + |T| = 6$, we have $|X_2| = |X_4| = 3$, and so $|S \cap C| = |A \cap T|, |B \cap T| = |D \cap S|$. By similar arguments as used in (1.1) we conclude that $A \cap D = \{z\}$. Since $|D| \geq 2$ and D is a connected subgraph of G, we have $D \cap S \neq \emptyset$. Since $A \cap C \neq \emptyset$, we find that X_1 is a vertex-cut of $G - uz$, then $|X_1| \geq 3$, and so $|S \cap C| \geq |B \cap T|, |A \cap T| \geq |D \cap S|$. By $|X_1| + |X_3| = |S| + |T| = 6$, we have $|X_3| \leq 3$. Since G is 4-connected, it follows that $B \cap D = \emptyset$. Noticing $|A \cap T| \geq |D \cap S|$, we have $|D \cap S| = |B \cap T| = 1$. Obviously, here D is a 1-edge-vertex-cut atom. Let $D = B', T = S', C = A', u = x', z = y'$. Then conclusion (ii) holds.

(3.) $x \in A \cap D$. By symmetry, analogous arguments as used in (1.) can prove the conclusion.

Subcase 1.2. $u = x$.

Then we have that $x \in A \cap C$, $y \in B \cap C$ or $B \cap T$. We distinguish the following Subcases to complete the proof of Subcase 1.2.

(1.) $y \in B \cap C$. Since $A \cap D \neq \emptyset$, X_2 is a vertex-cut of $G - xz$. Since G is 4-connected, we have $|X_2| \geq 3$. By similar arguments, we get that $|X_4| \geq 3$. Noticing $|X_2| + |X_4| = |S| + |T| = 6$, we find $|X_2| = |X_4| = 3$, and so $|S \cap C| = |A \cap T|, |B \cap T| = |D \cap S|$. First, we claim that $A \cap D = \{z\}$. Otherwise, $|A \cap D| \geq 2$. Let $A_1 = A \cap D, S_1 = X_2, B_1 = G - xz - S_1 - A_1$, then $(xz, S_1; A_1, B_1)$ is a separating group of G. Since $xz \in E_0$, A_1 is an E_0-edge-vertex-cut fragment contained in A, which contradicts the assumption that A is an E_0-edge-vertex-cut end-fragment. Hence $A \cap D = \{z\}$. Since $|D| \geq 2$ and D is a connected subgraph, we have $S \cap D \neq \emptyset$. If $|D \cap S| = |B \cap T| = 3$, then it is easy to see that $\{y, z\}$ is a 2-vertex-cut of G, a contradiction. So

$|D \cap S| = |B \cap T| \leq 2$.

(1.1.) $|B \cap T| = |D \cap S| = 2$. Since X_1 is a vertex-cut of $G - xy - xz$, we have $|X_1| \geq 2$. Note that $|S| = |T| = 3$ if and only if $|S \cap C| = |A \cap T| = 1, S \cap T = \emptyset$ hold. Here we claim that $A \cap C = \{x\}$. Otherwise, $|A \cap C| \geq 2$, then it is easy to see that $\{x\} \cup X_1$ is a 3-vertex-cut of G, a contradiction. Let $z = y', C = A', T = S', D = B'$. Then conclusion (iii) of the theorem holds.

(1.2.) $|B \cap T| = |D \cap S| = 1$. If $|S \cap T| = 2$, then $|C \cap S| = |A \cap T| = 0$. Here we have $|A| = 2$, which contradicts $|A| \geq 3$. Then we have $|S \cap T| \leq 1$. So $|X_3| \leq 3$, and hence $B \cap D = \emptyset$. Here D is a 1-edge-vertex-cut atom, and $|A \cap D| = |D \cap S| = 1$. Let $x = x', z = y', C = A', T = S', D = B'$. Then conclusion (ii) of theorem holds.

(2.) $y \in B \cap T$. By Theorem 2.1.2, we obtain $|C| = 2$. We claim that $C \cap S \neq \emptyset$. Otherwise, $S \cap C = \emptyset$. Since C is a connected subgraph, we have $B \cap C = \emptyset$. Then $C = A \cap C$, and C is an E_0-edge-vertex-cut fragment contained in A, which contradicts the assumption that A is an E_0-edge-vertex-cut end-fragment. So $|A \cap C| = |S \cap C| = 1$. Noticing $|S| = 3$, we have $|S \cap (D \cup T)| = 2$. If $B \cap T = \{y\}$, then $|X_3| = 3$, and so $B \cap D = \emptyset$. Here we find $B = \{y\}$, which contradicts $|B| \geq 2$. Hence $|B \cap T| \geq 2$. If $|B \cap T| = 3$. Then $T \cap (A \cup S) = \emptyset$ and $|X_1| = 1$. Here we get that $X_1 \cup \{y, z\}$ is a 3-vertex-cut of G, a contradiction. So $|B \cap T| = 2$, and $|A \cap C| = |S \cap C| = 1$. Let $x = y', z = x', C = B', T = S', D = A'$. So conclusion (ii) of theorem holds.

Case 2. There exists an edge $uz \in [A, S] \cap E_0$.

Obviously, $u \neq x$. Otherwise, $u = x$, and by Theorem 2.1.2, we obtain $|A| = 2$, which contradicts $|A| \geq 3$. Analogously, we consider the separating group $(uz, T; C, D)$ such that $u \in C, z \in D$. It is easy to see that $u \in A \cap C, z \in S \cap D$. The definition of X_1, X_2, X_3, X_4 is same as in Case 1. Here we distinguish subcases to complete the proof.

Subcase 2.1. $x \in A \cap C, y \in B \cap C$.

Since $B \cap C \neq \emptyset$, X_4 is a vertex-cut of $G - xy$, and so $|X_4| \geq 3$. Since $|X_2| + |X_4| = |S| + |T| = 6$, we have $|X_2| \leq 3$, and so $A \cap D = \emptyset$. If $A \cap T = \emptyset$, then $A = A \cap C$, and so $|A \cap C| \geq 3$. Since X_1 is a vertex-cut of $G - uz - xy$, then $|X_1| \geq 2$. Note that $D \cap S \neq \emptyset$ if and only if $|X_1| = |S \cap (C \cup T)| = 2$. We let $A_1 = A - \{u\}, S_1 = X_1 \cup \{u\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-edge-vertex-cut fragment contained in A, which contradicts that A is an E_0-edge-vertex-cut end-fragment. So $A \cap T \neq \emptyset$, and hence $|T \cap (B \cup S)| \leq 2$. If $S \cap D = \{z\}$. Then $|X_3| \leq 3$, and so $B \cap D = \emptyset$ and $D = \{z\}$, which contradicts $|D| \geq 2$. Hence $|D \cap S| \geq 2$, and then $|S \cap (C \cup T)| \leq 1$. Noticing that $|X_4| \geq 3$, we have $|B \cap T| \geq 2$, which implies $|B \cap T| = 2, |A \cap T| = 1$. So $S \cap T = \emptyset$. Here we have $|X_1| = 2$. Let $A_1 = A \cap C, S_1 = X_1 \cup \{z\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-edge-vertex-cut fragment contained in A, which contradicts the assumption that A is an E_0-edge-vertex-cut end-fragment. Therefore, Subcase 2.1 does not occur.

Subcase 2.2. $x \in A \cap C, y \in B \cap T$.

Since X_1 is a vertex-cut of $G - xy - uz$, we have $|X_1| \geq 2$. First, we show that $A \cap T = \emptyset$ holds. If $A \cap T \neq \emptyset$, then we claim that $|X_1| \geq 3$. Otherwise, $|X_1| = 2$. Obviously, $|A \cap C| \geq 2$. Let $A_1 = A \cap C, S_1 = X_1 \cup \{z\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-edge-vertex-cut fragment contained in A, which contradicts the assumption that A is an E_0-edge-vertex-cut end-fragment. So, $|X_1| \geq 3$, and $|C \cap S| \geq |B \cap T| \geq 1, |A \cap T| \geq |D \cap S| \geq 1$, which implies that $|B \cap T| = |D \cap S| = 1$. Since $|X_1| + |X_3| = 6$, we have $|X_3| \leq 3$, and so $B \cap D = \emptyset$. By $|D| \geq 2$, we know that $A \cap D \neq \emptyset$ and then we have $|X_2| \geq 4$ and $|X_4| \leq 2$. Hence $|B \cap C| = 0$, and $B = \{y\}$, which contradicts $|B| \geq 2$. Therefore, $A \cap T = \emptyset$. Since A is a connected subgraph, $A \cap D = \emptyset$, and so $|A| = |A \cap C| \geq 3$. Since $D \cap S \neq \emptyset$ and $|S| = 3$, we have $|X_1| = |S \cap (C \cup T)| = 2$. We let $A_1 = A - \{u\}, S_1 = X_1 \cup \{u\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-edge-vertex-cut fragment contained in A, a contradiction to the assumption. Therefore, Subcase 2.2 does not occur.

Subcase 2.3. $x \in A \cap T, y \in B \cap C$.

Since $B \cap C \neq \emptyset$, X_4 is a vertex-cut of $G - xy$, and then $|X_4| \geq 3$. Since $|X_2| + |X_4| = |S| + |T| = 6$, we have $|X_2| \leq 3$, and so $A \cap D = \emptyset$. Analogously, since X_1 is a vertex-cut of $G - uz$, we have $|X_1| \geq 3$. Noticing that $|X_1| + |X_3| = 6$, we find $|X_3| \leq 3$, and so $B \cap D = \emptyset$. Hence $|D| = |D \cap S| \geq 2$. Noticing that $|S| = 3$, we have $|S \cap (C \cup T)| \leq 1$. From $|X_4| \geq 3$, we get that $|B \cap T| \geq 2$. Then it is easy to see that $|A \cap T| = 1, S \cap T = \emptyset$. Obviously, $|X_1| \leq 2$, which contradicts the fact that $|X_1| \geq 3$. So, Subcase 2.3 does not occur.

Subcase 2.4. $x \in A \cap T, y \in B \cap D$.

Since X_1 is a vertex-cut of $G - uz$, we have $|X_1| \geq 3$. Similarly, $|X_3| \geq 3$. Since $|X_1| + |X_3| = |S| + |T| = 6$, we conclude $|X_1| = |X_3| = 3$. Then we get that $|A \cap T| = |D \cap S|, |C \cap S| = |B \cap T|$. First, we claim that $A \cap C = \{u\}$. Otherwise, $|A \cap C| \geq 2$. We let $A_1 = A \cap C, S_1 = X_1, B_1 = G - uz - S_1 - A_1$. Then $(uz, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-edge-vertex-cut fragment contained in A, which contradicts the assumption that A is an E_0-edge-vertex-cut end-fragment. So $A \cap C = \{u\}$. Since C is a connected subgraph and $|C| \geq 2$, we have $|C \cap S| = |B \cap T| \geq 1$. If $|C \cap S| = |B \cap T| = 2$, then $S \cap T = \emptyset, |A \cap T| = |D \cap S| = 1$. Clearly, we have $|X_2| = 2$. Then $A \cap D = \emptyset$. Here we have that $|A| = 2$, which contradicts $|A| \geq 3$. So $|S \cap C| = |B \cap T| = 1$, and C is a 1-edge-vertex-cut atom, and $|A \cap C| = |C \cap S| = 1$. Let $u = y', z = x', C = B', T = S', D = A'$. Then conclusion (ii) holds.

Subcase 2.5. $x \in A \cap D, y \in B \cap T$.

Since X_2 is a vertex-cut of $G - xy$, we have $|X_2| \geq 3$. By $|X_2| + |X_4| = |S| + |T| = 6$, we know $|X_4| \leq 3$. Then $B \cap C = \emptyset$. By similar arguments, we obtain $B \cap D = \emptyset$. Then we have that $|B| = |B \cap T| \geq 2$. Noticing that A is a connected subgraph, we have $A \cap T \neq \emptyset$, which implies $|A \cap T| = 1, |B \cap T| = 2$ and $S \cap T = \emptyset$. Since $|X_2| \geq 3$, we have that $|D \cap S| \geq 2$ and $|C \cap S| \leq 1$. Here we have that $|X_1| \leq 2$, which contradicts the assumption that X_1 is a vertex-cut of $G - uz$. So Subcase 2.5 does not occur.

Subcase 2.6. $x \in A \cap D, y \in B \cap D$.

Since X_2 is a vertex-cut of $G - xy$, we have $|X_2| \geq 3$. By $|X_2| + |X_4| = |S| + |T| = 6$, we know that $|X_4| \leq 3$, and so $B \cap C = \emptyset$. We claim that $C \cap S \neq \emptyset$. Otherwise, it is easy to see that C is an E_0-edge-vertex-cut fragment contained in A, contradicting the assumption that A is an E_0-edge-vertex-cut end-fragment. So $C \cap S \neq \emptyset$. Noticing that X_1 is a vertex-cut of $G - uz$, we get $|X_1| \geq 3$. Similarly, we have that $|X_3| \geq 3$. By $|X_1| + |X_3| = |S| + |T| = 6$, we know that $|X_1| = |X_3| = 3$, and so $|C \cap S| = |B \cap T| \geq 1, |A \cap T| = |D \cap S| \geq 1$. If $|C \cap S| = 2$, then $|A \cap T| = |D \cap S| = 1$, and so $|X_2| = 2$, a contradiction. Therefore, $|C \cap S| = |B \cap T| = 1$. We claim that $A \cap C = \{u\}$. Otherwise, if $|A \cap C| \geq 2$, we let $A_1 = A \cap C, S_1 = X_1, B_1 = G - uz - X_1 - A_1$. Then $(uz, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-edge-vertex-cut fragment, a contradiction. So $A \cap C = \{u\}$. Let $z = x', u = y', C = B', T = S', D = A'$. Therefore, conclusion (ii) holds. This completes the proof. $\square$

Lemma 5.1.2. *Let G be a 4-connected graph and $(xy, S; A, B)$ a separating group of G such that $x \in B, y \in A$. If there exists another edge $yz \in E_N(G)$ such that its corresponding separating group $(yz, S'; A', B')$ with $y \in A', z \in B'$ satisfy the following conditions:*

(i) *$A \cap A' = \{y\}, A \cap B' = \{z\}$, $A \cap S' = \{a\}, A' \cap S = \{b\}, B' \cap S = \{u, v\}$ such that $a, b, u, v \in G$.*

(ii) *$\{zu, zv\} \cap E_N(G) \neq \emptyset, ab \in E_N(G)$.*

Then au and av cannot belong to $E(G)$ simultaneously.

Proof. By contradiction. Assume $au, av \in E(G)$. Without loss of generality, we may assume that $zu \in E_N(G)$. Then consider the corresponding separating group $(zu, T_1; C_1, D_1)$ such that $z \in C_1, u \in D_1$. Then we have that $z \in C_1 \cap B', u \in B' \cap D_1$. Since $azua$ is a 3-cycle of G, we conclude $a \in T_1$, and so $a \in S' \cap T_1$. Let

$$Y_1 = (A' \cap T_1) \cup (S' \cap T_1) \cup (C_1 \cap S')$$

$$Y_2 = (C_1 \cap S') \cup (S' \cap T_1) \cup (B' \cap T_1)$$

$$Y_3 = (B' \cap T_1) \cup (S' \cap T_1) \cup (S' \cap D_1)$$

$$Y_4 = (D_1 \cap S') \cup (S' \cap T_1) \cup (A' \cap T_1)$$

Obviously, $y \in A' \cap C_1$ or $A' \cap T_1$. Next we distinguish the following cases to complete the proof.

Case 1. $y \in A' \cap C_1$.

Then Y_1 is a vertex-cut of $G - yz$. Since G is 4-connected, we have $|Y_1| \geq 3$. By similar arguments, we deduce that $|Y_3| \geq 3$. Since $|Y_1| + |Y_3| = |S'| + |T_1| = 6$, we obtain $|Y_1| = |Y_3| = 3$, and so $|A' \cap T_1| = |S' \cap D_1|, |S' \cap C_1| = |B' \cap T_1|$. Since $a \in S' \cap T_1$ and $ab \in E_N(G)$, by Theorem 2.1.4 we find $b \notin T_1$ and $b \notin S'$. Since $by \in E(G)$, we have that $b \in A' \cap C_1$. Since $zv \in E(G)$ and $v \in B'$, we know $v \in B' \cap (C_1 \cup T_1)$. Hence, $|A' \cap T_1| = |S' \cap D_1| = 0, 1$ or 2.

Now we distinguish the following Subcases for the value of $|A' \cap T_1|$ and $|D_1 \cap S'|$.

Subcase 1.1. $|A' \cap T_1| = |D_1 \cap S'| = 2$.

Noticing that $|T_1| = |S'| = 3$ and $a \in S' \cap T_1$, we have $|S' \cap C_1| = |B' \cap T_1| = 0$. Since $avza$ is a 3-cycle of G, we have that $v \in B' \cap C_1$, and so $|B' \cap C_1| \geq 2$. Then $\{a, z\}$ is a 2-vertex-cut of G, which contradicts that G is a 4-connected graph.

Subcase 1.2. $|A' \cap T_1| = |D_1 \cap S'| = 1$.

Then $|S' \cap T_1| \leq 2$. First, we claim that $B' \cap D_1 = \{u\}$. Otherwise, $|B' \cap D_1| \geq 2$. Since $\Gamma_G(a) = \{y, z, u, v, b\}$, by the foregoing arguments we have that $\Gamma_G(a) \cap (B' \cap D_1) = \{u\}$. Then $\{u\} \cup (Y_3 - \{a\})$ is a 3-vertex-cut of G, a contradiction. Hence, $D_1 \cap B' = \{u\}$. Let $D_1 \cap S' = \{u_1\}$. If $S \cap T_1 = \{a\}$, then $|Y_4| = 3$. Since G is 4-connected, we deduce $D_1 \cap A' = \emptyset$. Then $u_1 \in \Gamma_G(a)$. However, it is easy to see that $u_1 \notin \{y, z, b, u, v\}$ holds, a contradiction. Therefore, $|S' \cap T_1| = 2$. It is easy to see that $\Gamma_G(a) \cap (A' \cap D_1) = \emptyset$. If $A' \cap D_1 \neq \emptyset$, then $Y_4 - \{a\}$ is a 3-vertex-cut of G, a contradiction. If $A' \cap D_1 = \emptyset$, it is easy to see that $au_1 \in E(G)$ holds. However, $u_1 \notin \{b, u, v, y, z\}$, a contradiction.

Subcase 1.3. $|A' \cap T_1| = |D_1 \cap S'| = 0$.

Since D_1 is a connected subgraph of G, we find $A' \cap D_1 = \emptyset$. Since $|D_1| \geq 2$, we have that $|D_1 \cap B'| \geq 2$. By an analogous argument we can deduce that $\Gamma_G(a) \cap (D_1 \cap B') = \{u\}$. Since $|Y_3| = |T_1| = 3$, $\{u\} \cup (Y_3 - \{a\})$ is a 3-vertex-cut of G, a contradiction.

Case 2. $y \in A' \cap T_1$.

Since $yz \in E_N(G)$, by Theorem 2.1.2 we conclude that $|C_1| = 2$. Since C_1 is a connected subgraph of G, we find that $A' \cap C_1 = \emptyset$. If $S' \cap C_1 \neq \emptyset$. Since $|C_1| = 2$, we have that $|S' \cap C_1| = 1$. Note that $a \in S' \cap T_1$, we obtain $|D_1 \cap S'| \leq 1$. Since Y_3 is a vertex-cut of $G - zu$, it follows that $|Y_3| \geq 3$, and so $|B' \cap T_1| \geq 1$. Noticing $|T_1| = 3$, we have that $A' \cap T_1 = \{y\}$ and $|Y_4| = 3$. Since G is 4-connected, we deduce $A' \cap D_1 = \emptyset$, and therefore, we have that $A' = \{y\}$, which contradicts $|A'| \geq 2$. If $S' \cap C_1 = \emptyset$, then $|B' \cap C_1| = 2$. Since $A' \cap T_1 \neq \emptyset$, we have that $|Y_2| = |T_1 \cap (B' \cup S')| \leq 2$, and so $\{z\} \cup Y_2$ is a vertex-cut of G. However, $|\{z\} \cup Y_2| < 4$, a contradiction.

From all the above arguments we conclude that au, av cannot belong to $E(G)$ simultaneously. This completes the proof. $\square$

5.2 Removable Edges on the Hamilton Cycles

Before we present our main results, we first introduce the following definition:

Definition 5.2.1. Let G be a 4-connected graph, C a cycle of G, and $(xy, S; A, B)$ a 2-atom separating group. We say that C passes through a 2-atom if $x, y \in V(C)$.

The following lemma is used in the proof of the main result:

Lemma 5.2.1. *Let G be a 4-connected graph with $|G| \geq 7$, and let C be a cycle which does not pass through any 2-atom. Then there are at least two removable edges on C.*

Proof. By contradiction. Assume that C does not pass through any 2-atom of G, and there is at most one removable edge of G in C. Let $F = E(C) \cap E_R(G)$, then $|F| \leq 1$. Denote $E(C) - F$ by E_0. We consider the separating group $(uw, S'; A', B')$ such that $u \in A', w \in B'$ and $uw \in E_0$. By $|F| \leq 1$ we know that $(E(A') \cup [A', S']) \cap F = \emptyset$ or $(E(B') \cup [S', B']) \cap F = \emptyset$. Without loss of generality, we may assume $(E(A') \cup [A', S']) \cap F = \emptyset$. Since A' is an E_0-edge-vertex-cut fragment, A' must contain an E_0-edge-vertex-cut end-fragment as its subgraph, say A. Then we have $(E(A) \cup [A, S]) \cap F = \emptyset$. We consider a separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$. Since C does not pass through any 2-atom, we have $|A| \geq 3$. By Lemma 5.1.1, we know that one of the three conclusions of Lemma 5.1.1 holds. Here we discuss them as follows:

(1.) Since $(E(A) \cup [A, S]) \cap F = \emptyset$, obviously, the conclusion (i) does not hold.

(2.) By the assumption, we know that the conclusion (ii) of Lemma 5.1.1 does not hold.

(3.) If the conclusion (iii) of Lemma 5.1.1 holds, let $A \cap S' = \{w\}, B' \cap S = \{u, v\}, \Gamma_G(y') = \{w, u, v, x\}$. Since $|B'| \geq 3$, by Theorem 2.1.2 we have $y'w \in E_R(G)$. Noticing that C is a cycle and $(E(A) \cup [A, S]) \cap F = \emptyset$, we conclude $\{y'u, y'v\} \cap E_N(G) \neq \emptyset$. By Lemma 5.1.2, we have that wu, wv cannot belong to $E(G)$ simultaneously. Without loss of generality, we may assume that $wu \notin E(G)$. Let $A_0 = A - \{y'\}, S_0 = S \cup \{y'\} - \{u\}, B_0 = G - xy - S_0 - A_0$. Then A_0 is an E_0-edge-vertex-cut fragment contained in A, which contradicts the assumption that A is an E_0-edge-vertex-cut end-fragment. So, Conclusion (iii) does not hold. This complete the proof. $\square$

Now we present our main results.

Theorem 5.2.1. *Let G be a 4-connected graph with $|G| \geq 7$, and C a Hamilton cycle of G. If C does not pass through any 2-atom of G, then there are at least three removable edges on C.*

Proof. By contradiction. Assume that C does not pass through any 2-atom. If there exists a chord e of C such that $e \in E_N(G)$, then e separates the cycle C into two cycles C_1 and C_2. By Lemma 5.2.1 we know that both C_1 and C_2 have at least two removable edges, respectively, so that C has at least four removable edges. The conclusion holds. Now we assume that every chord of C is a removable edge of G, and C has at most two removable edges. By Lemma 5.2.1 we know that C just has two removable edges. Let $E(C) \cap E_R(G) = \{e_1, e_2\}$, then we have that $E(C) - \{e_1, e_2\} \subset E_N(G)$. We take $xy \in E(C) - \{e_1, e_2\}$ and its corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$. Let $E_0 = E(C) - \{e_1, e_2\}$, then A and B are E_0-edge-vertex-cut fragments. Since every E_0-edge-vertex-cut fragment contains an E_0-edge-vertex-cut end-fragment as its subgraph, without loss of generality, we may assume that A is an E_0-edge-vertex-cut end-fragment.

If $e_1, e_2 \in E[S]$. Since C is a cycle, we deduce that $xz \in E(C) \cap E_N(G) \cap (E(A) \cup [A, S])$ $(z \neq y)$. If $xz \in [A, S]$, then we find $|A| = 2$, a contradiction. So, we have that $xz \in E(A) \cap E_N(G)$. We consider the separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1, z \in B_1$. If $y \in B \cap S_1$, then we obtain $|A_1| = 2$, a contradiction. So $y \in A_1 \cap B$. Let $Y_1 = (A_1 \cap S) \cup (S \cap S_1) \cup (B \cap S_1)$, $Y_2 = (A \cap S_1) \cup (S \cap S_1) \cup (B_1 \cap S)$. Since Y_1 is a vertex-cut of $G - xy$ and Y_2 is a vertex-cut of $G - xz$, which implies that both $|Y_1| \geq 3$ and

$|Y_2| \geq 3$ holds. Since $|Y_1| + |Y_2| = 6$, we have that $|Y_1| = |Y_2| = 3$, and so $|A \cap S_1| = |A_1 \cap S|, |B_1 \cap S| = |B \cap S_1|$. We claim that $A \cap B_1 = \{z\}$ holds. Otherwise, (xz, Y_2) is a separating pair of G and $A \cap B_1$ is an E_0-edge-vertex-cut fragment contained in A. This contradicts the fact that A is an E_0-edge-vertex-cut end-fragment. Let $zz' \in E(C)$, then $zz' \in E_N(G)$. If $z' \in S_1$, by Theorem 2.1.2 we conclude $|B_1| = 2$, a contradiction. So $z' \in B_1$. Since $A \cap B' = \{z\}$, we have $z' \in S$. From $e_1, e_2 \in [S]$, we claim that $S \cap S_1 \neq \emptyset$. Otherwise, $|B_1 \cap S| = 3$ holds. Then $\{z, y\}$ is a 2-vertex-cut of G, a contradiction. So $|S \cap S_1| \geq 1$, and then $|B_1 \cap S| \leq 2$. If $|B_1 \cap S| = 2$, then $|S \cap S_1| = 1$, and $\{z, y\} \cup (S \cap S_1)$ is a 3-vertex-cut of G, a contradiction. So, $B_1 \cap S = \{z'\}$. If $|S \cap S_1| = 2$, then $A \cap S_1 = \emptyset = A_1 \cap S$. We claim that $|A| = 2$. Otherwise, $|A| \geq 3$ holds, and so $|A \cap A_1| \geq 2$. Then $\{x\} \cup (S \cap S_1)$ is a 3-vertex-cut of G, a contradiction. So, $|A| = 2$, a contradiction. Hence, $|S \cap S_1| = 1$, and $|A_1 \cap S| = 1$. Here we have that $|B \cap S_1| + |S \cap S_1| + |B_1 \cap S| = 3$, so $B \cap B_1 = \emptyset$, and $|B_1| = 2$, a contradiction.

Therefore, next we may assume that either $(E(A) \cup [A, S]) \cap \{e_1, e_2\} \neq \emptyset$ or $(E(B) \cup [B, S]) \cap \{e_1, e_2\} \neq \emptyset$ holds. If $(E(B) \cup [B, S]) \cap \{e_1, e_2\} = \emptyset$, then B must contain an E_0-edge-vertex-cut end-fragment as its subgraph, say B_0, and consider its corresponding separating group $(x_0y_0, S_0; A_0, B_0)$ such that $x_0 \in A_0, y_0 \in B_0$. It is easy to see that $|(E(B_0) \cup [S_0, B_0]) \cap E(C) \cap E_R(G)| \leq 1$ holds. So, without loss of generality, we may assume $|(E(A) \cup [A, S]) \cap E(C) \cap E_R(G)| \leq 1$. Let $E_1 = E(A) \cup [A, S]$.

Since C does not pass through any 2-atom, we have $|A| \geq 3$. By Lemma 5.1.1 we know that one of the conclusions (i),(ii) or (iii) holds. Next we will discuss them respectively.

Case 1. Conclusion (i) holds.

Let $t \in A - \{x\}$. Since $t \in C$, and $d(t) \geq 4$, we have $|E_1 \cap E(C) \cap E_R(G)| \geq 2$, which contradicts $|E_1 \cap E(C) \cap E_R(G)| \leq 1$.

Case 2. Obviously, according to the assumption, the conclusion (ii) does not hold.

Case 3. Conclusion (iii) holds.

We consider a separating group of G as in conclusion (iii) of Lemma 5.1.1. Let $B' \cap S = \{b, c\}$, $A' \cap S = \{a\}, A \cap S' = \{d\}$. First, we claim that $bd, cd \in E(G)$. Otherwise, we may assume $bd \notin E(G)$. Let $A_1 = A - \{y'\}, S_1 = S - \{b\} \cup \{y'\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group and A_1 is an E_0-edge-vertex-cut fragment contained in A, a contradiction. It is easy to see that $(ad, \{x, b, c\})$ is a separating pair of G, so $ad \in E_N(G)$. Since every chord of C is removable, we find $ad \in E(C)$. From Lemma 5.1.2 we know that $y'b, y'c \in E_R(G)$. Since $|B'| \geq 3$, by Theorem 2.1.2 we know that $dy' \in E_R(G)$.

(1.) $dy' \in E(C)$. Then we have $by', cy' \notin E(C)$. Since $ad \in E(C)$, we deduce $bd, cd \notin E(C)$. We let P_1 denote the path going from vertex y' to b on C which does not pass through vertex d, and P_2 going from vertex d to b on C and does not pass through vertex y'. Let $C_1 = P_1 + by'$ and $C_2 = P_2 + bd$. Obviously, if $bd \in E_R(G)$, then neither C_1 nor C_2 passes through any 2-atom. By Lemma 5.2.1 we know that there are at least two removable edges on C_1 and C_2, respectively. If $bd \in E_N(G)$, and if C_2 pass through a 2-atom, then only the 2-atom separating group $(bd, S_0; A_0, B_0)$ through which bd passes happens. We may assume that $|A_0| = 2$. Let $A_0 = \{b, w\}$, it is easy to see that $y' \in S_0$, then only $w = c$ holds. However, $cd \in E(G)$, a contradiction. Noticing that $dy' \in E_R(G) \cap E(C)$, so we have that in this case Theorem holds.

(2.) $dy' \notin E(C)$. Without loss of generality, we may assume that $by' \in E(C)$. Next we distinguish the following cases to complete the proof:

(2.1.) $bd \in E(C)$. By $|E_1 \cap E(C) \cap E_R(G)| \leq 1$ we know that $bd \in E_N(G)$. We let P_1 denote the path going from vertex y' to c on cycle C which does not pass through vertex b, and P_2 going from vertex b to c on cycle C which does not pass through vertex y'. Let $C_1 = P_1 + cy'$ and $C_2 = P_2 + bc$, then neither C_1 nor C_2 passes through any 2-atom of G. By Lemma 5.2.1 we know that there are at least two removable edges on C_1 and C_2, respectively. Noticing $by' \in E_R(G) \cap E(C)$, the conclusion holds.

(2.2.) $cd \in E(C)$. Then $bd \notin E(C)$. We let P_1 denote the path going from vertex y' to d on cycle C which does not pass through vertex b, and P_2 going from vertex b to d on cycle C and does not pass through vertex y'. Let $C_1 = P_1 + dy'$ and $C_2 = P_2 + bd$. By similar arguments we deduce that neither C_1 nor C_2 passes through any 2-atom of G. By Lemma 5.2.1 we know that there are at least two removable edges on C_1 and C_2, respectively. Since $by' \in E_R(G) \cap E(C)$, we know that in this case theorem holds. This completes the proof. $\square$

Lemma 5.2.2. *Let G be a 4-connected graph with $|G| \geq 7$, C a cycle which exactly contains one inner vertex of some maximal l-bi-fan H and does not pass through any other subgraph belonging to $\Re$. Then there are at least two removable edges on C.*

Proof. By contradiction. Assume that there is at most one removable edge on C. By Theorem 4.2.2 we know there is precisely one removable edges on C. Let $E(C) \cap E_R(G) = \{e\} = F$. Let H be a maximal l-bi-fan as defined in Definition 1.2.2. Based on the assumption $|V(C) \cap (V(H) - \{x_1, x_{l+3}\})| = 1$ and $|E(C) \cap E_R(G)| = 1$, it can be checked easily that either $x_2 \in C$ or $x_{l+2} \in C$ holds. Without loss of generality, we may assume $x_{l+2} \in V(C)$, and $e = ax_{l+2}$. By letting $S' = \{a, b, x_{l+3}\}, e' = x_2x_1, B' = \{x_2, \cdots, x_{l+2}\}, A' = G - e' - S' - B'$, then $(e', S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of the maximal l-bi-fan. From the assumption we have that A' does not contain any inner vertex of subgraph belonging to $\Re$. Let $E_0 = E(C) - \{e\}$, then A' is an E_0-edge-vertex-cut fragment. Obviously, A' contains an E_0-edge-vertex-cut end-fragment as its subgraph, say A. It is easy to see that A does not contain any inner vertex of H and $(E(A) \cup [A, S]) \cap F = \emptyset$. We consider a separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$. Next we will consider the following cases for $|A|$.

(1.) $|A| = 2$. Then either A is a 1-edge-vertex-cut atom or a 2-edge-vertex-cut atom, say $A = \{x, z\}$. Let $S = \{a, b, c\}$.

(1.1.) A is a 2-edge-vertex-cut atom. Since $xy \in E(C)$ and C is a cycle of G, we have that $\{xa, xb, xc, xz\} \cap E(C) \neq \emptyset$. From Lemma 3.1.2 we know that $\{xa, xb, xc, xz\} \subset E_R(G)$, which contradicts $(E(A) \cup [A, S]) \cap F = \emptyset$.

(1.2.) A is a 1-edge-vertex-cut atom. By noticing that C is a cycle of G and $(E(A) \cup [A, S]) \cap F = \emptyset$, we have that $\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$. From Corollary 3.1.1 we know that x is an inner vertex of one of the subgraphs of G belonging to $\Re$, a contradiction.

(2.) $|A| \geq 3$. Then from Lemma 5.1.1 we know that one of the conclusion (i),(ii) or (iii) of lemma holds.

(2.1.) If conclusion (i) holds. Since C is a cycle, and $(E(A) \cup [A, S]) \cap F = \emptyset$, a contradiction is obtained.

(2.2.) If conclusion (ii) holds, we let $A \cap B' = \{y'\}, \Gamma_G(y') = \{x', a', b', z'\}$. Noticing that C is a cycle, and $(E(A) \cup [A, S]) \cap F = \emptyset$, then $\{a'y', by', y'z'\} \cap E_N(G) \neq \emptyset$. From Corollary 3.1.1 we know that vertex y' is an inner vertex of a subgraph belonging to $\Re$, a contradiction.

(2.3.) If conclusion (iii) holds. By noticing that C is a cycle, Lemma 5.1.2 yields a contradiction.

Based on the above arguments we know that the assumption is not true. Therefore, the lemma holds. This completes the proof of lemma.□

Theorem 5.2.2. *Let G be a 4-connected Hamilton graph with $|G| \geq 7$, C a Hamilton cycle of G. Then if C passes through only one subgraph (excluding maximal l-belt or l-co-belt) belonging to $\Re$, but doesn't pass through any maximal l-belt or l-co-belt, then there are at least two removable edges on C.*

Proof. By contradiction. Assume that C passes through only one subgraph belonging to $\Re$ and doesn't pass through any maximal l-belt or l-co-belt, but there are at most one removable edge on cycle C. Then we will discuss the cases as follows:

Case 1. C passes through *helm* H.

Let H be defined as in Definition 1.2.1. Since C is a Hamilton cycle, it is easy to see that there is at least one removable edge on C. Let $F = E_R(G) \cap E(C)$, then $|F| = 1$. Without loss of generality, we may assume that $F = \{x_3x_4\}$. According to the assumption, we have that $E(C) - \{x_3x_4\} \subset E_N(G)$. By letting $e = x_1v_1, S' = \{v_2, v_3, v_4\}, B' = \{a, x_1, x_2, x_3, x_4\}, A' = G - e - S' - B'$, then $(e, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of H with $E(C) \cap (E(A') \cup [A', S']) \cap E_R(G) = \emptyset$. Let $E_0 = E(C) - \{x_3x_4\}$, then A' is an E_0-edge-vertex-cut fragment of G such that it does not contain any inner vertex of H. Obviously, A' contains an E_0-edge-vertex-cut end-fragment as its subgraph, say A. It is easy to see that A does not contain any inner vertex of H and $(E(A) \cup [A, S]) \cap F = \emptyset$. And we take a separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$.

Now we can apply similar arguments as used in Lemma 5.2.2 to get that the Case 1 yields a contradiction.

Here we give an example to show that in this case the lower bound is sharp. See figure 5.1.

Example 5.2.1. Let H be a helm as in Definition 1.2.1, $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$, $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, noindent x_3v_3, x_4v_4\}$.

Let $L = H - \{v_1, v_2, v_3, v_4\}$, L' a copy of L such that $V(L') = \{a', x_1', x_2', x_3', x_4'\}$. Now we construct a graph G as follows: $V(G) = V(L) \cup V(L')$, and join vertices x_1 to x_1', x_2 to x_2', x_3 to x_3', x_4 to x_4', x_2' to x_4', respectively. $(x_1x_1', \{x_2, x_3, x_4\})$ is a separating pair of G, hence $x_1x_1' \in E_N(G)$. By symmetry, we have that $x_2x_2', x_3x_3', x_4x_4' \in E_N(G)$. It is easy to see that $(a'x_1', \{x_2', x_4', x_3\}), (a'x_3', \{x_2', x_4', x_1\})$ are separating pairs of G. So we have that $a'x_1', a'x_3' \in$

$E_N(G)$. Let $C = x_1x_1'a'x_3'x_3x_4x_4'x_2'x_2ax_1$, then C is a Hamilton cycle through which passes one helm and contains precisely two removable edges $x_3x_4, x_2'x_4'$.

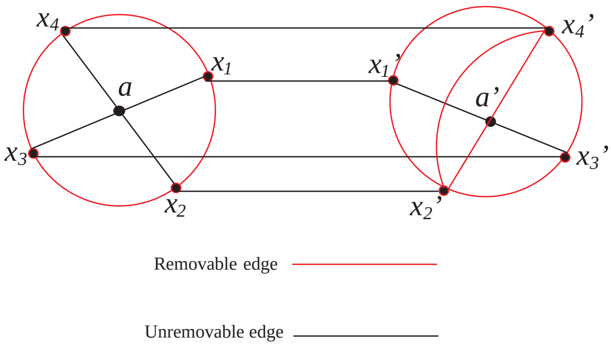

Figure 5.1:

Case 2. H is a W'-framework as defined in Definition 1.2.6.

Let $F = E(C) \cap E_R(G)$. Since C is a Hamilton cycle and by assumption $|E(C) \cap E_R(G)| = 1$, it can be checked easily that $y_1y_2 \in E(C)$ holds. By letting $S = \{x_1, x_3, y_4\}, B = \{x_2, y_2, y_3\}, A = G - y_1y_2 - S - B$, then $(y_1y_2, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H. From the assumption we can get that $F \cap (E(A) \cup [A, S]) = \emptyset$. We apply similar arguments as used in Case 1 to prove that the conclusion holds.

Here we give an example to show that in this case the lower bound is sharp. See figure 5.2.

Example 5.2.2. Let H be a W'-framework as in Definition 1.2.6 with $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$. Let L' be a graph as defined in Example 5.2.1, $V(L') = \{a', x_1', x_2', x_3', x_4'\}$. We construct a graph G as follows: Let $V(G) = V(H) - \{y_1, y_4\} \cup V(L')$, $E(G) = (E(H) - \{y_1y_2, y_3y_4\}) \cup E(L') \cup \{x_1x_1', y_2x_4', y_3x_3', x_2'x_3, x_1x_3', x_1'x_3'\}$.

It can be checked easily that G is a 4-connected graph. First, we have that $(x_1x_1', \{x_2', x_3', x_4'\}), (x_2'x_3, \{x_1', x_3', x_4'\}), (y_2x_4', \{x_1', x_2', x_3'\}), (y_3x_3', \{x_1, x_3, y_2\}),$

$(a'x_4', \{x_1', x_3', x_3\}), (a'x_2', \{y_2, x_1', x_3'\})$ are separating pairs of G, so $x_1x_1', x_2'x_3$, $y_2x_4', y_3x_3', a'x_4', a'x_2' \in E_N(G)$.

Let $C = x_1x_2x_3x_2'a'x_4'y_2y_3x_3'x_1'x_1$. Then C is a Hamilton cycle through which passes precisely one W'-framework and contains two removable edges $y_2y_3, x_1'x_3'$.

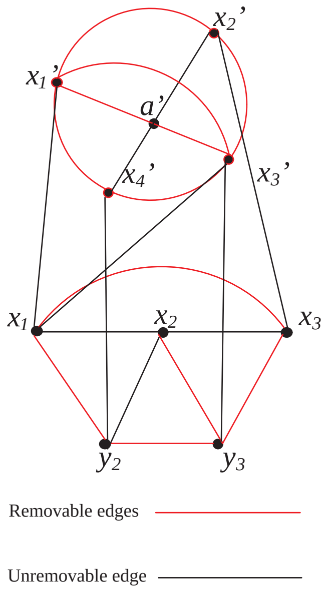

Figure 5.2:

Case 3. H is a W-framework defined as in Definition 1.2.5.

Since C is a Hamilton cycle and assumption $|E(C) \cap E_R(G)| = 1$, it is easy to see that $y_1y_2 \in E(C)$ and $y_2y_3 \in E(C) \cap E_R(G)$. By letting $S = \{x_1, x_3, y_4\}, B = \{x_2, y_2, y_3\}, A = G - y_1y_2 - S - B$, then $(y_1y_2, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the W-framework, and $E_R(G) \cap E(C) \cap (E(A) \cup [A, S]) = \emptyset$. We apply similar arguments as used in Case 1 to prove that the conclusion holds.

Here we give an example to show that in this case the lower bound is sharp. See figure 5.3

Example 5.2.3. Let H be a W-framework as in Definition 1.2.5, L a graph such that $V(L) = \{a', x'_1, x'_2, x'_3, x'_4\}$, $E(L) = \{a'x'_1, a'x'_2, a'x'_3, a'x'_4, x'_1x'_2, x'_2x'_3, x'_3x'_4, x'_4x'_1, x'_1x'_3\}$.

Now we construct a graph G as follows: Let $V(G) = V(L) \cup V(H) - \{y_1, y_2\}$, $E(G) = E(L) \cup E(H) - \{y_1y_2, y_3y_4\} + \{x_1x'_1, x'_2x_3, y_3x'_3, y_2x'_4, x_1x'_3, x'_1x_3\}$. It can be checked easily that G is a 4-connected graph. We have that $(x'_2x_3, \{x'_1, x'_3, x'_4\})$, $(y_3x'_3, \{x_1, x_3, y_2\})$, $(y_2x'_4, \{x'_1, x'_2, x'_3\})$, $(x'_1x_1, \{y_2, x_3, x'_3\})$, $(a'x'_4, \{x'_1, x'_3, x_3\})$, $(a'x'_2, \{y_2, x'_1, x'_3\})$ are separating pairs of G. Let $C = x_1x_2x_3x'_2a'x'_4y_2y_3x'_3x'_1x_1$. Obviously, C is a Hamilton cycle which contains two removable edges $x'_1x'_3$, y_2y_3.

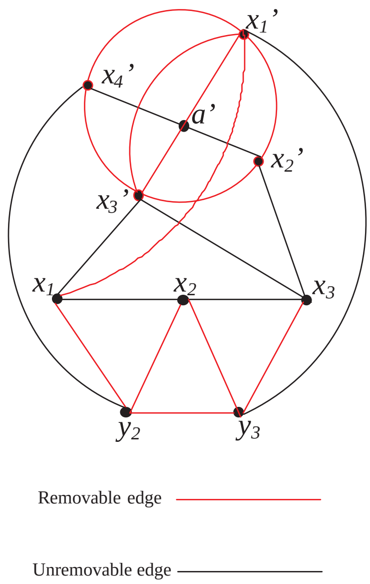

Figure 5.3:

Case 4. H is a maximal l-bi-fan which is defined as in Definition 1.2.2.

Here we have that either $|E(C) \cap \{ax_2, ax_3, \cdots, ax_{l+2}\}| \leq 1$ or $|E(C) \cap \{bx_2, bx_3, \cdots, bx_{l+2}\}| \leq 1$ holds. Without loss of generality, we may assume $|E(C) \cap \{ax_2, ax_3, \cdots, ax_{l+2}\}| \leq 1$. Next we distinguish the following two subcases to complete the proof of Case4.

Subcase 4.1. $\{x_1x_2, x_2x_3, \cdots, x_{l+2}x_{l+3}\} \subset E(C)$.

Let $C = x_1x_2\cdots x_{l+2}x_{l+3}\cdots vau\cdots x_1\}$. We let P_1 denote the path going from a to x_2 on C through which passes vertex u, and P_2 going from x_{l+2} to a on C through which passes vertex v. Then $C_1 = P_1 + ax_2$ and $C_2 = P_2 + ax_{l+2}$ are two cycles which contain just one inner vertex of l-bi-fan and don't pass through any other subgraph belonging to $\Re$. By Lemma 5.2.2 we know that C_1 and C_2 contain at least two removable edges respectively, so there are at least two removable edges on C.

Subcase 4.2. For some $i \in \{2, \cdots, l+2\}$, we have that either $\{ax_i, x_ix_{i+1}, \cdots, x_{l+2}x_{l+3}\} \subset E(C)$ or $\{bx_i, x_ix_{i+1}, \cdots, x_{l+2}x_{l+3}\} \subset E(C)$ holds.

Without loss of generality, we may assume $\{ax_i, x_ix_{i+1}, \cdots, x_{l+2}x_{l+3}\} \subset E(C)$. From $|E(C) \cap \{ax_2, \cdots, ax_{l+2}\}| \leq 1$, we deduce $(\{ax_2, \cdots, ax_{i+2}\} - \{ax_i\}) \cap E(C) = \emptyset$. Then it is easy to see that only $i = 2$ holds. Let $C = ax_2x_3\cdots x_{l+2}x_{l+3}\cdots ua$. Let P denote the path going from x_{l+2} to a on C and passes through vertex u. Then $C_1 = P + ax_{l+2}$ is a cycle of G which passes through just one inner vertex of l-bi-fan and doesn't pass through any other subgraph belonging to $\Re$. From Lemma 5.2.2 we know that C_1 contains at least two removable edges, and so P contains at least one removable edge. Since $ax_2 \in E(C) - E(C_1)$, therefore, there are at least two removable edges on C.

Example 5.2.4. Here we give an example to show that in this case the lower bound is sharp. See figure 5.4.

Let H be an l-bi-fan ($l \geq 2$) as defined in Definition 1.2.2, L a graph such as $V(L) = \{a', x_1', x_2', x_3', x_4'\}$, $E(L) = \{a'x_1', a'x_2', a'x_3', a'x_4', x_1'x_2', x_2'x_3', x_3'x_4', x_4'x_1', x_1'x_3'\}$.

Now we construct a graph G as follows: First, we identify the vertex x_1 with x_1', x_{l+3} with x_4', respectively. Then join the vertices a and x_2', b and x_3', a and b. Obviously, G is a 4-connected graph. Similar arguments as used in Example 5.2.2 can show that $bx_3', ax_2', x_{l+2}x_4', x_2x_1', a'x_4', a'x_2' \in E_N(G)$. Let

$C = ax'_2a'x'_4x_{l+2}x_{l+1}\cdots x_2x'_1x'_3ba$, then C is a Hamilton cycle through which passes two removable edges $x'_1x'_3$ and ab.

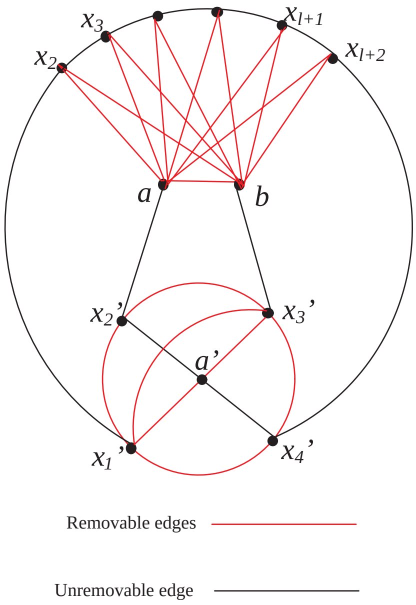

Figure 5.4:

Noticing that in this conclusion, we do not discuss the case that C pass through only one l-belt or l-co-belt. In fact, even if for a Hamilton cycle C through which passes only one maximal l-belt or maximal l-co-belt, Theorem 3.2.2 can not be improved. We can give the examples to show that. See figure 5.5.

Example 5.2.5. (1.) Let H be a maximal l-belt as in Definition 1.2.3, and L a graph such that $V(L) = \{a', x'_1, x'_2, x'_3, x'_4\}$, $E(L) = \{a'x'_1, a'x'_2, a'x'_3, a'x'_4, x'_1x'_2, x'_2x'_3, x'_3x'_4, x'_4x'_1, x'_1x'_3\}$. Now we construct a graph G as follows: First , we identify the vertex x_1 with x'_1, y_{l+2} with x'_3, respectively. Then we connect vertices y_1 and x'_2, x_{l+2} and x'_4, y_1 and x_{l+2}, respectively. We denote the resulting graph by G, it can be easily checked that G is a 4-connected graph. Similar arguments can lead to the fact that $x'_1x_2, x'_2y_1, x'_3y_{l+1}, x'_4x_{l+2} \in E_N(G)$. Let $C = x'_1x_2x_3\cdots x_{l+2}x'_4a'x'_2y_1y_2\cdots y_{l+1}x'_3x'_1$, then C is a Hamilton cycle which contains only one removable edge $x'_1x'_3$.

(2.) Let H be a maximal l-co-belt as in Definition 1.2.4, and let L be defined as in (1.), Now we construct a graph G as follows: First , we identify the vertex x_{l+3} with x_4', x_1 with x_1', respectively. Then connect vertices y_1 and x_2', y_1 and y_{l+2}, x_3' and y_{l+2}, respectively. It is easy to see that G is a 4-connected graph. Similar arguments can be used as in (1.) to show that $C = x_1'x_2x_3\cdots x_{l+2}x_4'a'x_2'y_1y_2\cdots y_{l+2}x_3'x_1'$ is a Hamilton cycle which contains only one removable edge $x_1'x_3'$.

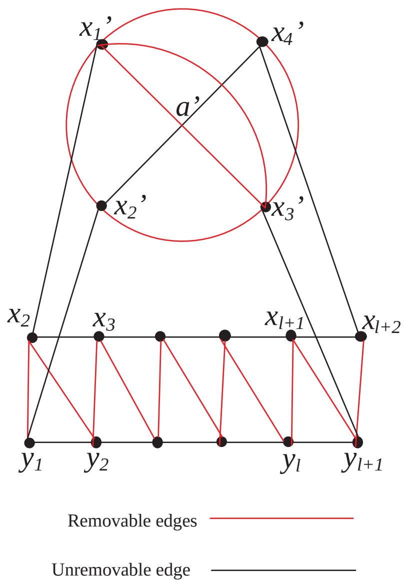

Figure 5.5:

Theorem 5.2.3. *Let G be a 4-connected Hamilton graph with $|G| \geq 7$, and let C be a Hamilton cycle of G. Then the following conclusions hold: If C pass through just two subgraphs belonging to $\Re$ excluding maximal l-belt or l-co-belt, then there is at least one removable edge on C.*

Proof. By contradiction. Assume that C passes through just two subgraphs (excluding maximal l-belt or l-co-belt) belonging to $\Re$ and does not pass through any maximal l-belt or l-co-belt, and there is no removable edge on cycle C. Then we will distinguish the following cases to complete the proof.

Case 1. C passes through any two subgraphs as following: W'-framework,

W-framework and *helm*.

Then it is easy to see that the conclusion holds.

Case 2. G contains just two maximal l-bi-fan.

Let $H_1 = (a, b; x_1, x_2, \cdots, x_{l+3}\}$ and $H_2 = \{a', b'; y_1, y_2, \cdots, y_{t+3}\}$ be just two maximal l-bi-fan and t-bi-fan which C passes through. Assume that $E(C) \cap E_R(G) = \emptyset$. Then we can get that $E(C) \cap \{ax_2, ax_3, \cdots, ax_{l+2}, bx_2, bx_3, \cdots, bx_{l+2}, a'y_2, a'y_3, \cdots, a'y_{t+2}, b'y_2, \cdots, b'y_{t+2}\} = \emptyset$. Consequence we have $\{x_1x_2, x_2x_3, \cdots, x_{l+2}x_{l+3}, y_1y_2, \cdots, y_{t+2}y_{t+3}\} \subset E(C)$ holds. Without loss of generality, we may assume $C = x_1x_2 \cdots x_{l+3} \cdots y_1y_2 \cdots y_{t+2}y_{t+3} \cdots x_1$. Let P_1 denote the path going from x_{l+1} to y_1 on cycle C and does not pass through y_2, let P_2 denote the path going from y_{t+1} to x_1 on C and does not pass through x_2. Since $d(a) \geq 5, d(b) \geq 5, d(a') \geq 5, d(b') \geq 5$, vertices a, b, a', b are not inner vertices of l-bi-fan. So, we have that $a \in P_1$ or $a \in P_2$. Without loss of generality, we may assume $a \in P_2$. Let P_3 denote the path from a to x_2 on C and passes through x_1. Then $P_3 + ax_2$ is a cycle which does pass through neither H_1 nor H_2. From Lemma 5.2.2 we know that $P_3 + ax_2$ contains at least two removable edges, and so P_3 contains at least one removable edges. Therefore, theorem holds.

The following example shows that in this case the lower bound is sharp. See figure 5.6.

Example 5.2.6. Let L be a graph such that $V(L) = \{a_1, y_1, y_2, y_3, y_4\}$, $E(L) = \{a_1y_1, a_1y_2, a_1y_3, a_1y_4, y_1y_2, y_2y_3, y_3y_4, y_4y_1, y_1y_3\}$, let H and H' be two bi-fans defined as in Definition 1.2.2 with $V(H) = \{a, b; x_1, x_2, \cdots, x_{l+2}, x_{l+3}\}$, $V(H') = \{a', b'; x'_1, x'_2, \cdots, x'_{t+2}, x'_{t+3}\}$. Now we construct the graph G as follows:

First: We delete the vertices $x_1, x_{l+3}, x'_1, x'_{t+3}$ from H and H' respectively. Second: Identify the vertex a with a', and join the vertices a and y_1, a and y_2, x_2 and b', b and x'_2, x_{l+2} and x'_{t+2}, b' and y_3, b and y_4, respectively. It can be easily checked that G is a 4-connected graph, and $(bx'_2, \{a, b', x'_{t+2}\})$,

$(b'x_2, \{a, b, x_{l+2}\}), (x_{l+2}x'_{t+2}, \{a, b, x_2\}), (a_1y_4, \{a, y_1, y_3\}), (a_1y_2, \{y_1, y_3, b\}), (ay_1, \{y_2, y_3, y_4\}), (ay_2, \{y_1, y_3, y_4\})$ are separating pairs of G. Let $C = ay_1y_3b'x_2x_3 \cdots x_{l+2}x'_{t+2}x'_{t+1} \cdots x'_2by_4a_1y_2a$, it can be easily checked that C is a Hamilton cycle with only one removable edge y_1y_3. See figure 5.5.

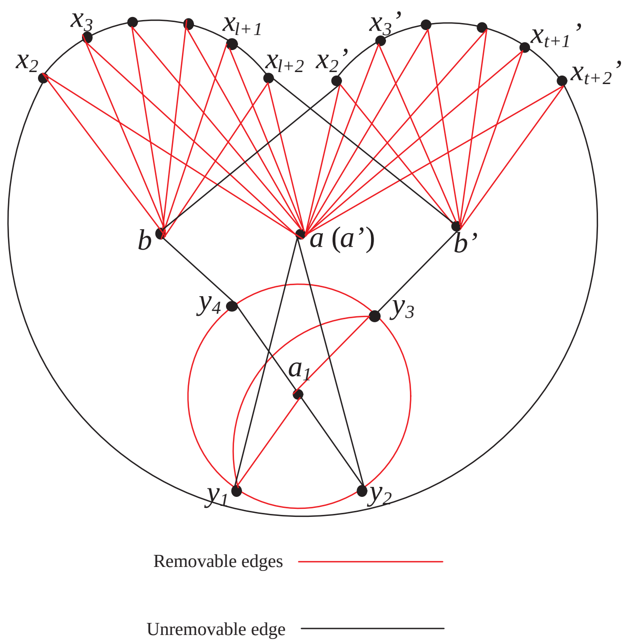

Figure 5.6:

This completes the proof.□

Chapter 6

The Number of Removable Edges in 4-Connected Graphs

In this chapter we prove that every 4-connected graph of order at least six (excluding the 2-cyclic graph of order six) has at least $(4|G|+16)/7$ removable edges. We also give a structural characterization of 4-connected graphs for which the lower bound is sharp.

6.1 Some Subgraphs and their Properties

For convenience, some special notations are introduced.

By L_1 we denote the maximal 1-belt such that $V(L_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and $E(L_1) = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, y_1x_2, x_2y_2, y_2x_3\}$. We say that x_2x_3, y_1y_2 are inner edges of L_1.

By L_2 we denote the maximal 2-belt such that $V(L_2) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$ and $E(L_2) = \{x_1x_2, x_2x_3, x_3x_4, y_1y_2, y_2y_3, y_3y_4, y_1x_2, x_2y_2, y_2x_3, x_3y_3, y_3x_4\}$. We say that $x_2x_3, x_3x_4, y_1y_2, y_2y_3$ are inner edges of L_2.

By L_1' we denote the maximal 1-co-belt such that $V(L_1') = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}$ and $E(L_1') = \{x_1x_2, x_2x_3, x_3x_4, y_1y_2, y_2y_3, y_1x_2, x_2y_2, y_2x_3, x_3y_3\}$. We say that x_2x_3, y_1y_2, y_2y_3 are inner edges of L_1'.

By L'_2 we denote the maximal 2-co-belt such that $V(L'_2) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4\}$ and $E(L'_2) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5, y_1y_2, y_2y_3, y_3y_4, y_1x_2, x_2y_2, y_2x_3, x_3y_3, y_3x_4, x_4y_4\}$. We say that $x_2x_3, x_3x_4, y_1y_2, y_2y_3, y_3y_4$ are inner edges of L'_2.

By F we denote the maximal 1-bi-fan such that $V(F) = \{a, b, x_1, x_2, x_3, x_4\}$ and $E(F) = \{x_1x_2, x_2x_3, x_3x_4, ax_2, ax_3, bx_2, bx_3\}$. We say that x_2x_3 is an inner edge of F.

By W we denote the W-framework such that $V(W) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(W) = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. We say that x_1x_2, x_2x_3 are inner edges of W.

By W' we denote the W'-framework such that $V(W') = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(W') = \{x_1x_2, x_2x_3, x_1x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. We say that x_1x_2, x_2x_3, x_2y_2 are inner edges of W'.

By H we denote the helm such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$. We say that the edges ax_i for $i = 1, 2, 3, 4$ are inner edges of H.

Let Υ denote the set of special graphs defined above, so $\Upsilon = \{L_1, L_2, L'_1, L'_2, F, W, W', H\}$. Then we first prove the following useful observation on the members of set of Υ.

Lemma 6.1.1. *There is no common inner edge between any two different subgraphs of G in Υ.*

Proof. By contradiction. Suppose that there are two different subgraphs K and K' of G in Υ that have a common inner edge. Then we discuss the following cases.

(1.) K is a maximal 1-belt L_1. Then x_2x_3 and y_1y_2 are the inner edges of K.

Without loss of generality, we may assume that x_2x_3 is also an inner edge of K'. We discuss the following subcases for K'.

(1.1.) K' is a maximal 1-belt. Let $V(K') = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ and $E(K') = \{u_1u_2, u_2u_3, v_1v_2, v_2v_3, v_1u_2, u_2v_2, v_2u_3\}$, and let the inner edges of K' be u_2u_3, v_1v_2. If $x_2x_3 = u_2u_3$, then we have either $x_2 = u_2, x_3 = u_3$ or $x_2 = u_3, x_3 = u_2$. If $x_2 = u_2, x_3 = u_3$, then $K = L_1 = K'$. If $x_2 = u_3, x_3 = u_2$, then we have either $d(x_3) = 4$ and $x_3y_3 \in E(G)$ or we have $d(y_1) = 4$ and $x_1y_1 \in E(G)$. However, this contradicts that K is a maximal 1-belt.

(1.2.) Similar arguments show that K' is not a maximal 1-co-belt, a maximal 2-belt or a maximal 2-co-belt.

(1.3.) K' is a maximal 1-bi-fan. Then we have that either $x_3y_1 \in E(G)$ or $x_1x_3 \in E(G)$. If $x_1x_3 \in E(G)$, then from the definition of the maximal 1-bi-fan, we have that $x_1x_2 \in E_R(G)$, which contradicts the definition of the maximal 1-belt K. If $x_3y_1 \in E(G)$, since $y_1y_2 \in E_N(G)$, we consider the corresponding separating group $(y_1y_2, S; A, B)$ such that $y_1 \in A, y_2 \in B$. Since $y_1y_2x_2y_1, y_1y_2x_3y_1$ are 3-cycles of G, we have that $x_2x_3 \in E([S])$. By Theorem 2.1.4 we have that $x_2x_3 \in E_R(G)$, which contradicts the definition of the maximal 1-belt K. Therefore, any inner edge of a maximal 1-belt can not be an inner edge of any maximal 1-bi-fan, and vice versa.

(1.4.) K' is a W-framework or a W'-framework. Then we have that $y_1y_2 \in E_R(G)$, which contradicts the definition of the maximal 1-belt K. Hence, any inner edge of a maximal 1-belt can not be an inner edge of any W-framework or W'-framework, and vice versa.

(1.5.) K' is a helm. Then either x_2 or x_3 is incident with four unremovable edges in G. Obviously, this is impossible since x_2x_3 is an inner edge of the maximal 1-belt K. Therefore, any inner edge of a maximal 1-belt can not be an inner edge of any helm, and vice versa.

(2.) K is a maximal 2-belt L_2. Without loss of generality, we may assume that x_2x_3 is a common inner edge of K and K'. We distinguish the following subcases for K'.

(2.1.) K' is also a maximal 2-belt. Let $V(K') = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ and $E(K') = \{u_1u_2, u_2u_3, u_3u_4, v_1v_2, v_2v_3, v_3v_4, v_1u_2, u_2v_2, v_2u_3, u_3v_3, v_3u_4\}$, and let $u_2u_3, u_3u_4, v_1v_2, v_2v_3$ be the inner edges of K'. If $x_2x_3 = u_2u_3$, then one of the following holds: (i) $K = L_2 = K'$; (ii) $d(y_1) = 4$ and $x_1y_1 \in E(G)$, which contradicts that K is a maximal 2-belt. If $x_2x_3 = v_1v_2$, it is easy to see that $u_1v_1 \in E(G)$ and $d(v_1) = 4$, which contradicts that K' is a maximal 2-belt. By symmetry, for the other cases, we may apply similar arguments to show that the conclusion holds.

(2.2.) Since a maximal 1-co-belt is a subgraph of a maximal 2-belt, it is easy to see that x_2x_3 or y_1y_2 is not an inner edge of a maximal 1-co-belt. Otherwise, it contradicts the definition of the maximal 1-co-belt. Similarly, a maximal 2-belt and a maximal 2-co-belt do not have any common inner edge.

(2.3.) Obviously, it is impossible that an inner edge of a maximal 2-belt is an inner edge of the following subgraphs: maximal 1-bi-fan, W-framework, W'-framework or helm. And vice versa.

(3.) K is a maximal 2-co-belt. It is easy to see that an argument similar to that used in (2.) can be applied to deduce contradictions.

(4.) K is a maximal 1-bi-fan. If K' is also a maximal 1-bi-fan F', it is easy to see that this is true only if $F = F'$ holds. Obviously, it is impossible that the inner edge x_2x_3 of H is an inner edge of the following subgraphs: W-framework, W'-framework or helm.

(5.) K is a W-framework, or a W'-framework, or a helm. Obviously, no matter whatever K' is, we always can deduce contradictions. The details are omitted. $\square$

6.2 Some Preliminary Results

In order to obtain a sharp lower bound on the number of removable edges in a 4-connected graph, we first prove the following preliminary results.

Theorem 6.2.1. *Let G be a 4-connected graph and F a maximal l-bi-fan of G with $l \geq 2$. Then there exists an edge e' in F such that $e' \in E_R(G)$ and $e_R(G) \geq e_R(G \ominus e') + 1$.*

Proof. Let F be defined as in Definition 1.2.2. First, we claim that $d(a) \geq 5, d(b) \geq 5$. Otherwise, we may assume that $d(a) = 4$ and let $\Gamma_G(a) = \{x_2, x_3, x_4, v\}$. We claim that $v \neq b$. Otherwise, $\{x_2, x_4, b\}$ is a 3-vertex-cut of G, a contradiction. Let $A = \{a, x_3\}, S = \{x_2, x_4, v\}, e = bx_3, B = G-e-A-S$. Then $(bx_3, S; A, B)$ is a separating group of G, and therefore $bx_3 \in E_N(G)$, which contradicts that F is an l-bi-fan.

Let $e' = ax_3, H = G \ominus e'$. Next we show that for any edge $e \neq x_2x_4$ in H, if $e \in E_R(H)$, then $e \in E_R(G)$.

By contradiction. Assume that there exists an edge $e \in E_R(H)$, but $e \in E_N(G)$. Let $e = xy$. Since $xy \in E_N(G)$, by Theorem 2.1.1 we can consider the corresponding separating group $(e, T; C, D)$ such that $x \in C, y \in D$. We distinguish the following cases to prove the conclusion.

Case 1. $a, x_3 \in T$.

Since $d(x_3) = 4$ and $ax_3 \in E(G)$, we have that $|\Gamma_G(x_3) \cap C| = 1$ or $|\Gamma_G(x_3) \cap D| = 1$. Without loss of generality, we may assume that $|\Gamma_G(x_3) \cap C| = 1$. Let $\Gamma_G(x_3) \cap C = \{v_1\}, T = \{a, x_3, w\}$. If $|C| \geq 3$, let $T' = \{a, v_1, w\}$, $C' = C - \{v_1\}$ and $D' = H - xy - T' - C'$. We claim that $v_1 \neq x$. Otherwise, we have that $\{a, w, v_1\}$ is a 3-vertex-cut of G, which contradicts that G is 4-connected. It is easy to see that $(e, T'; C', D')$ is a separating group of

H, and therefore $e \in E_N(H)$, a contradiction. If $|C| = 2$, then $v_1x \in E(G)$. Since $d(b) \geq 5$ and $v_1 \neq b$, we have $v_1 \in \{x_2, x_4\}$. If $v_1 = x_2$, then $x = x_1$. Since $\Gamma_G(x_2) = \{b, x_1, x_3, a\}$, we have that $w = b$ and $\Gamma_G(x_1) = \{a, b, x_2, y\}$. Obviously, $\{ax_1, bx_1\} \subset E_R(G)$ and $x_1y \in E_N(G)$, which contradicts the definition of a maximal l-bi-fan of G. If $v_1 = x_4$, then $w = b$, and therefore $\Gamma_G(x) = \{a, b, x_4, y\}$, and so $x = x_5$. Let $C' = \{x_4, x\}, e = xy, T' = \{a, b, x_2\}, D' = H - e - C' - T'$. Then $(e, T'; C', D')$ is a separating group of H, and so $e \in E_N(H)$, which contradicts that $e \in E_R(H)$.

Case 2. $a \in T, x_3 \in C$.

So, $\Gamma_G(x_3) = \{a, b, x_2, x_4\}$. If $|C| \geq 3$, then it is easy to see that $(e, T; C - \{x_3\}, D)$ is a separating group of H, and hence $e \in E_N(H)$, which contradicts that $e \in E_R(H)$. Therefore, $|C| = 2$, and so $x \in \Gamma_G(x_3)$. If $x = b$, then $T = \{a, x_2, x_4\}$, $\Gamma_G(b) = \{a, x_2, x_3, x_4, y\}, \Gamma_G(x_2) \cap D = \{x_1\}$. Since $x_1x_4 \notin E(G)$ and $x_1 \neq y$, we have that $|D| \geq 3$. Let $T' = \{a, x_1, x_4\}, D' = D - \{x_1\}, C' = H - xy - T' - D'$. Then $(xy, T'; C', D')$ is a separating group of H, and so $e \in E_N(H)$, a contradiction. If $x = x_2$, then $y = x_1$. Obviously, if we let $e = x_2x_1, C' = \{x_2, x_4\}, T' = \{a, b, x_5\}, D' = H - e - C' - T'$, then $(e, T'; C', D')$ is a separating group of H, and so $x_2x_1 \in E_N(H)$, a contradiction. If $x = x_4$, then $y = x_5$. Let $C' = \{x_2, x_4\}, T' = \{a, b, x_1\}, D' = H - x_4x_5 - T' - C'$. Then $(x_4x_5, T'; C', D')$ is a separating group of H, and so $x_4x_5 \in E_N(H)$, a contradiction.

Case 3. $a \in C, x_3 \in T$.

If $|C| = 2$, then $a = x$, and so $C - \{a\} = \{x_2\}$ or $C - \{a\} = \{x_4\}$. If $C - \{a\} = \{x_2\}$, then $b \in T$. Since $x_3x_4 \in E_N(G)$, by Theorem 2.1.4 we have $x_4 \notin T$. If $x_4 \in D - \{y\}$, then $ax_4 \notin E(G)$, a contradiction. If $C - \{a\} = \{x_4\}$, by similar arguments can leads to a contradiction, and therefore $|C| \geq 3$. Since $a \in C$, we have that $x_2, x_4 \in C \cup T$. Noticing that $\Gamma_G(x_3) \cap D \neq \emptyset$, we have $b \in D$, and so $x_2, x_4 \in T$. But then $\{x_2, x_4, x\}$ is a 3-vertex-cut of H, a contradiction.

Case 4. $a, x_3 \in C$.

Obviously, $|C| \geq 3$, and similar arguments lead to a contradiction.

Based on the above arguments, we know that if $e \in E_R(H)$ and $e \neq x_2x_4$, then $e \in E_R(G)$. Noticing that $ax_3, bx_3 \in E_R(G)$, but $ax_3, bx_3 \notin E(H)$, we prove that $e_R(G) \geq e_R(G \ominus e) + 1$.$\square$

Theorem 6.2.2. *Let G be a 4-connected graph and L a maximal l-belt of G with $l \geq 3$. Then there exists an edge e' in $E(G)$ such that $e_R(G) \geq e_R(G \ominus e') + 2$.*

Proof. Let L be defined as in Definition 1.2.3. Consider $e' = x_3y_3$ and let $H = G \ominus e'$. We delete three removable edges y_2x_3, y_3x_3, y_3x_4 from G and add three edges y_2x_4, x_2x_4, y_2y_4 to get H. Let $A' = \{y_2, x_2\}, e_1 = y_2y_4, S' = \{x_1, y_1, x_4\}$ and $B' = G - e_1 - S' - A'$. Then $(e_1, S'; A', B')$ is a separating group of H, and hence $y_2y_4 \in E_N(H)$. By similar arguments we can show $x_2x_4 \in E_N(H)$. It remains to show that for any $e \in E(H)$ and $e \neq y_2x_4$, if $e \in E_R(H)$, then $e \in E_R(G)$. We prove this by contradiction. Assume that there exists an edge $e \in E_R(H)$, but $e \in E_N(G)$. Let $e = xy$. By Theorem 2.1.1 we consider the corresponding separating group $(e, S; A, B)$ such that $x \in A, y \in B$. Next we distinguish the following cases to complete the proof.

Case 1. $x_3, y_3 \in S$.

Let $S = \{x_3, y_3, w\}, w \in G$ and $U = \{x_2, x_4, y_2, y_4\}$. Noticing that $\Gamma_G(x_3) = \{x_2, x_4, y_2, y_3\}$ and $\Gamma_G(y_3) = \{x_3, x_4, y_2, y_4\}$, we claim that $|A \cap U| = 2 = |B \cap U|$. Otherwise, we may assume that $|A \cap U| = 1$. Let $A \cap U = \{v_1\}$, then $\{x, v_1, w\}$ is a 3-vertex-cut of G, which contradicts that G is 4-connected. If $|A| = 3$, since $l \geq 3$, obviously we have that $|G| \geq 10$, and so $|B| \geq 4$. Let $B \cap U = \{v_1, v_2\}$. If we let $S_1 = \{v_1, v_2, w\}$, $B_1 = B - \{v_1, v_2\}, A_1 = H - e - S_1 - B_1$, then $(e, S_1; A_1, B_1)$ is a separating group of H, and so $e \in E_N(H)$, a contradiction. If $|A| \geq 4$, let $A \cap U = \{u_1, u_2\}, S_1 = \{u_1, u_2, w\}$, $A_1 = A - \{u_1, u_2\}, B_1 = H - e - S_1 - A_1$. Then $(e, S_1; A_1, B_1)$ is a separating group of H, and so $e \in E_N(H)$, which contradicts the assumption.

Case 2. $x_3 \in A, y_3 \in S$.

Subcase 2.1. $|A| = 2$.

Then $x \in \Gamma_G(x_3)$. If $x = x_2$, then $S = \{y_2, y_3, x_4\}$. Since $x_2y_3, x_2x_4 \notin E(G)$, we have that $d(x_2) < 4$, a contradiction. If $x = x_4$, similar arguments leads to $d(x_4) < 4$, a contradiction. If $x = y_2$. Then $y = y_1$. Let $A_1 = \{y_2, x_4\}, e = y_1y_2, S_1 = \{x_2, x_5, y_4\}$, $B_1 = H - e - A_1 - S_1$, then $(e, S_1; A_1, B_1)$ is a separating group of H, and so $e \in E_N(H)$, which contradicts the assumption.

Subcase 2.2. $|A| \geq 3$.

Since $x_3 \in A$, it is easy to see that $B \cap \Gamma_G(y_3) = \{y_4\}$. If $|B| \geq 3$, let $B_1 = B - \{y_4\}, S_1 = \{y_4\} \cup S - \{y_3\}, A_1 = H - e - S_1 - B_1$. Then $(e, S_1; A_1, B_1)$ is a separating group of H, and so $e \in E_N(H)$. If $|B| = 2$, since $\Gamma_G(y_4) = \{y_3, y_5, x_4, x_5\}$, we have $y \in \{x_4, x_5, y_5\}$. If $y = x_4$, then this is true only if $x = x_3$ holds, a contradiction. If $y = x_5$, since $y_3x_5 \notin E(G)$, we have $d(x_5) = 4$ and $S = \{y_3, y_5, x_4\}$. Let $A_1 = A - \{y_2\}, S_1 = \{y_2, y_5, x_4\}$, $B_1 = H - e - S_1 - A_1$. Then $(e, S_1; A_1, B_1)$ is a separating group of H, and hence $e \in E_N(H)$. If $y = y_5$, then $S = \{x_4, x_5, y_3\}$. Note that $y_3y_5, x_4y_5 \notin E(G)$. So, $d(y_5) < 4$, a contradiction.

To sum up, from the above arguments we know that in Case 2 we always have $e \in E_N(H)$.

Case 3. $x_3 \in S, y_3 \in A$.

By symmetry, arguments analogous to that used in Case 2 can lead to that $e \in E_N(H)$.

Case 4. $x_3, y_3 \in A$.

If $|A| \geq 4$. Obviously, $e \in E_N(H)$, which contradicts the assumption. So, $|A| \leq 3$. Obviously, $x_3 \neq x, y_3 \neq x$. Therefore, we have that $|A| = 3$. Since A is a connected subgraph of G, we may assume that $x_3x \in E(G)$. If $x = x_4$, then $xy = x_4x_5$. Let $S_1 = \{y_1, y_4, x_2\}, A_1 = \{y_2, x_4\}, B_1 = H - e - S_1 - A_1$, then $(e, S_1; A_1, B_1)$ is a separating group of H, and so $e \in E_N(H)$. If $x = y_2$, then $y = y_1$. Let $e = y_2y_1, A_1 = \{y_2, x_4\}, S_1 = \{x_2, x_5, y_4\}, B_1 = H - e - S_1 - A_1$, then $(e, S_1; A_1, B_1)$ is a separating group of H, and so $e \in E_N(H)$. If $x = x_2$, then $S = \{y_2, y_4, x_4\}$. It is easy to see that $d(x_2) < 4$, a contradiction.

Based on the above arguments, we have that $E_R(H) \subseteq E_R(G)$ except for the edge y_2x_4. Noticing that $y_2x_3, x_3y_3, x_4y_3 \in E_R(G)$, we prove that $e_R(G) \geq e_R(G \ominus e) + 2$. □

A 4-connected graph G is said to have property $(\star)$ if there does not exist any edge $xy \in E_R(G)$ such that both $d(x) \geq 5$ and $d(y) \geq 5$.

Theorem 6.2.3. *Let G be a 4-connected graph with property $(\star)$, $|G| \geq 8$, and C' be a cycle of G. If C' does not contain any removable edges of G, then G has one of the following structures as its subgraph: l-belt, l-bi-fan ($l \geq 1$), W-framework, W'-framework or helm, such that it intersects C' at of its some inner edge(s).*

Proof. For every edge $e = xy$ in C', by Theorem 2.1.1 there exists a separating group $(e, S; A, B)$ of G, in which we always choose A and B such that $min\{|A|, |B|\}$ is as small as possible. Without loss of generality, we may assume $|A| \leq |B|$ such that $y \in A, x \in B$. Then we consider $f = yz \in E(C'), z \neq x$, and its corresponding separating group $(f, T; C, D)$ such that $y \in C, z \in D$ in G. Let

$$X_1 = (S \cap C) \cup (S \cap T) \cup (A \cap T)$$

$$X_2 = (A \cap T) \cup (S \cap T) \cup (S \cap D)$$

$$X_3 = (S \cap D) \cup (S \cap T) \cup (B \cap T)$$

$$X_4 = (B \cap T) \cup (S \cap T) \cup (S \cap C)$$

It is easy to see that the edge $e = xy$ is the unique edge connecting A and B, and the edge $f = yz$ is the unique edge connecting C and D. So $x \notin D, z \notin B$. Since X_1 is a vertex-cut of $G - yx - yz$ and G is 4-connected, we have $|X_1| \geq 2$.

Next we will distinguish the following cases to complete the proof.

Case 1. $x \in B \cap C, z \in D \cap S$.

By Theorem 2.1.2 we have $|A| = 2$. Since $A \cap C \neq \emptyset$ and A is a connected subgraph of G, we have $A \cap D = \emptyset$, and so $|A \cap T| \leq 1$. If $|A \cap T| = 0$, then $|A \cap C| = 2$. Since $S \cap D \neq \emptyset$, by noticing that $|S| = 3$, we get $|X_1| = |(S \cap C) \cup (S \cap T)| \leq 2$, and hence $X_1 \cup \{y\}$ is a vertex-cut of G. However, $|X_1 \cup \{y\}| < 4$, which contradicts that G is 4-connected. Therefore, $|A \cap T| = 1, A \cap C = \{y\}$. Since X_4 is a vertex-cut of $G - xy$, we have $|X_4| \geq 3$, and hence $|S \cap C| \geq |A \cap T| = 1, |B \cap T| \geq |S \cap D| \geq 1$. So $S \cap T = \emptyset$ or $|S \cap T| = 1$. We claim that $S \cap T = \emptyset$. Otherwise, if $|S \cap T| = 1$, then $|X_3| = 3$, and so $B \cap D = \emptyset$. Since $A \cap D = \emptyset$, it is easy to see that $D = D \cap S = \{z\}$, which contradicts $|D| \geq 2$, and thus $S \cap T = \emptyset$. Noticing that $|T| = 3$, we have $|B \cap T| = 2$. If $|S \cap C| = 2$, then $|S \cap D| = 1$. By similar arguments we get that $D = \{z\}$, which contradicts $|D| \geq 2$. Therefore, $|C \cap S| = 1$, and so $|D \cap S| = 2$.

Let $A \cap T = \{a\}, S \cap C = \{b\}, S \cap D = \{z, c\}$. It is easy to see that $\Gamma_G(y) = \{x, z, a, b\}, \Gamma_G(a) = \{y, z, b, c\}$. Next we show that $ay, az, by \in E_R(G)$ by contradiction.

(1.) Assume that $ay \in E_N(G)$ and we consider a separating group $(ay, U; A', B')$ such that $a \in A', y \in B'$. Since $ayza, abya$ are 3-cycles of G, we have that $z, b \in U$. Since $yz \in E_N(G)$, by Theorem 2.1.2 we get $|B'| = 2$. Let $B' = \{v_1, y\}$, then byv_1b is a 3-cycle of G and $v_1 \neq a$. It is easy to see that this is true only if $v_1 = x$ holds. However, $xz \notin E(G)$, and so $d(x) < 4$, a contradiction.

(2.) Assume that $az \in E_N(G)$ and we consider its separating group $(az, U; A', B')$ such that $a \in A', z \in B'$ in G. Since $ayza$ is a 3-cycle of G, we have $y \in U$. Since $yz \in E_N(G)$, from Theorem 2.1.2 we have that $|B'| = 2$. Let $B' = \{z, v_1\}$, then yzv_1y is a 3-cycle of G and $v_1 \neq a$, which is impossible. Therefore, $az \in E_R(G)$.

(3.) Assume that $by \in E_N(G)$. First, let $A' = C \cap (B \cup S), S' = \{y\} \cup (B \cap T)$, $B' = G - ab - A' - S'$, then $(ab, S'; A', B')$ is a separating group of G, and hence $ab \in E_N(G)$. Since $by \in E_N(G)$, we consider its separating group $(by, U; A', B')$ such that $b \in A', y \in B'$. Since $abya$ is a 3-cycle of G, we have that $a \in S'$. Since $ab \in E_N(G)$, by Theorem 2.1.2 we have that $|A'| = 2$. Let $A' = \{b, v_1\}$. Then abv_1a is a 3-cycle of G and $v_1 \neq y$, which is impossible in G, and therefore, we have $by \in E_R(G)$.

Let $A' = \{a, y\}, S' = \{b, z, x\}$, $B' = G - ac - S' - A'$. Then $(ac, S'; A', B')$ is a separating group of G, and so $ac \in E_N(G)$. It is easy to see that $(ab, B \cap T \cup \{y\})$ is a separating pair of G, so $ab \in E_N(G)$.

Obviously, yz is an inner edge of an l-belt or l-co-belt with $l \geq 1$, and so the conclusion holds.

Case 2. $z \in S \cap D, x \in B \cap T$.

By Theorem 2.1.2 we have $|A| = |C| = 2$. Since A and C are two connected subgraphs of G, we have $A \cap D = \emptyset = B \cap C$. First, we claim that $|A \cap C| = 1$. Otherwise, $|A \cap C| = 2$, and so $A \cap T = \emptyset = S \cap C$. Since $B \cap T \neq \emptyset \neq S \cap D$, we have $|X_1| = |S \cap T| \leq 2$, and so $X_1 \cup \{y\}$ is a vertex-cut of G. However, $|X_1 \cup \{y\}| < 4$, which contradicts that G is 4-connected. Therefore, $|A \cap T| = 1, |S \cap C| = 1$. Second, we claim that $S \cap T = \emptyset$. Otherwise, $|S \cap T| = 1$. Then $|X_3| = 3$, and so $B \cap D = \emptyset$. Hence, $D = D \cap S = \{z\}$, which contradicts $|D| \geq 2$. Therefore, we have $|B \cap T| = |S \cap D| = 2$.

Let $A \cap T = \{a\}, S \cap C = \{b\}, D \cap S = \{z, v\}, B \cap T = \{x, u\}$. Then $\Gamma_G(y) = \{x, z, a, b\}, \Gamma_G(a) = \{x, z, b, v\}, \Gamma_G(b) = \{x, y, a, u\}$.

Next we show $az \in E_R(G)$. By contradiction, assume that $az \in E_N(G)$ and we consider the corresponding separating group $(az, U; A', B')$ such that $a \in A', z \in B'$. Since $azya$ is a 3-cycle of G, we have $y \in U$. Since $yz \in E_N(G)$, by Theorem 2.1.2 we have $|B'| = 2$. Let $B' = \{z, v_1\}$. Then yzv_1y is a 3-cycle of G and $v_1 \neq a$, and so this is true if only if $v_1 = x$ holds. Since $bx \in E(G)$, we have $b \in U$. Then $(U - \{y\}) \cup \{a\}$ is a 3-vertex-cut of G, a contradiction. Therefore, $az \in E_R(G)$ holds. By symmetry, we obtain $bx \in E_R(G)$. Let $A' = \{a, y\}$, $S' = \{x, z, b\}, B' = G - av - S' - A'$. Then $(av, S'; A', B')$ is a separating group of G, and so $av \in E_N(G)$. By similar arguments we can lead to $bu \in E_N(G)$.

Now we discuss the following subcases.

Subcase 2.1. $xz \notin E(G)$.

First we show that $ay, by \in E_R(G)$. By contradiction, we assume that $ay \in E_N(G)$ and consider its separating group $(ay, U; A', B')$ such that $a \in A', y \in B'$. Since $ayza$ is a 3-cycle of G, we have $z \in U$. Since $yz \in E_N(G)$, by Theorem 2.1.2 we have $|B'| = 2$. Let $B' = \{y, v_1\}$, then yzv_1y is a 3-cycle of G. Obviously, $v_1 \neq a$. Note that $xz \notin E(G)$, and so $v_1 \neq x$, which is impossible in G. Therefore, we have $ay \in E_R(G)$. By symmetry, we have $by \in E_R(G)$. It is easy to see that if $ab \in E_N(G)$, then G contains an l-belt or an l-co-belt with $l \geq 1$ such that yz is its an inner edge. If $ab \in E_R(G)$, then G contains a W-framework such that yz is its an inner edge. Therefore, the conclusion holds.

Subcase 2.2. $xz \in E(G)$.

Since $xy, yz \in E_N(G)$, by Corollary 2.1.3 we have $xz \in E_R(G)$. Since G has property $(\star)$, we have either $d(x) = 4$ or $d(z) = 4$.

Subcase 2.2.1. $d(x) = 4, d(z) \geq 5$.

Let $\Gamma_G(x) = \{y, z, b, w\}$. Since $|G| \geq 8$, we have $B \cap D \neq \emptyset$, and so $w \in B \cap D$. Let $A' = \{x, y\}$, $U = \{w, z, b\}, B' = G - ay - U - A'$. Then $(ay, U; A', B')$ is a separating group of G, and so $ay \in E_N(G)$. We claim that $ab \in E_R(G)$. Otherwise, $ab \in E_N(G)$. Then we consider a separating group $(ay, T_1; C_1, D_1)$ of G such that $a \in C_1, y \in D_1$. Obviously, $z, b \in T_1$. Since $ab, yz \in E_N(G)$, by Theorem 2.1.2 we have $|C_1| = |D_1| = 2$, which contradicts $|G| \geq 8$, and so $ab \in E_R(G)$. We claim that $by \in E_R(G)$. Otherwise, $by \in E_N(G)$, we consider its separating group $(by, T_1; C_1, D_1)$ such that $b \in C_1, y \in D_1$. Since $byxb$ is a 3-cycle of G, we have $x \in T_1$. Since $xy \in E_N(G)$, by Theorem 2.1.2 we have $|D_1| = 2$. Let $D_1 = \{y, v_1\}$, then yxv_1y is a 3-cycle of G, and hence this is true only if $v_1 = z$ holds. However, $d(v_1) = 4$, which contradicts $d(z) \geq 5$. Therefore, $by \in E_R(G)$. Obviously, we have that xy, yz are inner edges of a W'-framework in G. The conclusion holds.

Subcase 2.2.2. $d(x) \geq 5, d(z) = 4$.

By symmetry, by arguments similar to that used in Subcase 2.2.1 we can get that the conclusion holds.

Subcase 2.2.3. $d(x) = d(z) = 4$.

Let $\Gamma_G(x) = \{y, z, b, w\}$. Let $A' = \{x, y\}$, $U = \{w, z, b\}, B' = G - ay - U - A'$, then $(ay, U; A', B')$ is a separating group of G, and so $ay \in E_N(G)$. By symmetry, we have $by \in E_N(G)$. Since $xy, yz \in E_N(G)$, from Corollary 2.1.3 we have that $ab, bx, xz, za \in E_R(G)$. Obviously, G contains a helm as a subgraph such that xy, yz are its inner edges. Therefore, the conclusion holds.

Case 3. $z \in A \cap D, x \in B \cap T$.

By Theorem 2.1.2 we have $|C| = 2$. Since $|A| \leq |C|$, we have $|A| = 2$, and hence $A = \{y, z\}$, $A \cap T = \emptyset$. Since $A \cap D \neq \emptyset$, we have $|X_2| \geq 3$. Noticing that $|S| = 3$, we have $|A \cap T| \geq |S \cap C|$, and so $|S \cap C| = 0$. Since C is a connected subgraph of G and $|C| = 2$. Since $A = \{y, z\}$, we have $A \cap C = \{y\}$. Therefore, $C \cap S \neq \emptyset$, a contradiction. So, Case 3 does not occur.

Case 4. $z \in A \cap D, x \in B \cap C$.

So, $A \cap D \neq \emptyset \neq B \cap C$, and therefore $|X_2| \geq 3, |X_4| \geq 3$. Since $|X_2| + |X_4| = |S| + |T| = 6$, we have $|X_2| = |X_4| = 3$, and so $|A \cap T| = |S \cap C|, |B \cap T| = |S \cap D|$. First, we claim that $A \cap D = \{z\}$. Otherwise, $|A \cap D| \geq 2$. Let $U' = X_2, A' = A \cap D, B' = G - yz - U' - A'$. Then $(yz, U'; A', B')$ is a separating group of G, and $yz \in E(C')$, $|A'| < |A|$, which contradicts that $|A|$ is as small as possible. Therefore, $A \cap D = \{z\}$. Since D is a connected subgraph of G and $|D| \geq 2$, we have $D \cap S \neq \emptyset \neq B \cap T$, and so $|S \cap T| \leq 2$. If $|S \cap T| = 1$, we claim that $S \cap C \neq \emptyset \neq A \cap T$. Otherwise, $|X_1| = 1$. Obviously, $|A \cap C| \geq 2$, and so $\{y\} \cup (S \cap T)$ is a 2-vertex-cut of G, a contradiction. Therefore, $|S \cap C| = |A \cap T| = 1, |D \cap S| = |B \cap T| = 1$, and hence $|X_3| = 3$. Then we have that $B \cap D = \emptyset$ and $|D| = 2$. However, $|A| \geq 3$. Then $|D| < |A|$, which contradicts that $|A|$ is as small as possible. Therefore, $|S \cap T| = 0$ or $|S \cap T| = 2$.

Next we show that $|S \cap T| \neq 0$. Assume that $|S \cap T| = 0$. Then we have $|B \cap T| = |S \cap D| = 2$ and $|A \cap T| = |S \cap C| = 1$. We claim that $A \cap C = \{y\}$. Otherwise, $|A \cap C| \geq 2$. Then $X_1 \cup \{y\}$ is a 3-vertex-cut of G, which contradicts that G is 4-connected, and so $d(y) = 4$. Let $A \cap T = \{a\}, S \cap C = \{b\}, S \cap D = \{u, v\}$. First, let $A' = \{a, z\}, S' = \{y\} \cup (S \cap D), B' = G - ab - S' - A'$. Then $(ab, S'; A', B')$ is a separating group of G, and so $ab \in E_N(G)$. Second, we claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, we consider the separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $y \in S'$. Since $yz \in E_N(G)$, by Theorem 2.1.2 we have $|B'| = 2$, say $B' = \{z, v_1\}$. Then zv_1yz is a 3-cycle of G and $v_1 \neq a$, which is impossible to hold, so $az \in E_R(G)$. Since C' is a cycle of G, we have $\{zu, zv\} \cap E_N(G) \neq \emptyset$. By Lemma 5.1.2 we have that au, av can not belong to $E(G)$ simultaneously. Without loss of generality, we may assume that $au \notin E(G)$. Let $S' = (S - \{u\}) \cup \{z\}$, $A' = A - \{z\}, B' = B \cup \{u\}$. Then

$(xy, S'; A', B')$ is a separating group of G, and $|A'| < |A|$, which contradicts that $|A|$ is as small as possible. Therefore, $S \cap T \neq \emptyset$, and so $|S \cap T| = 2$. Then, we have that $|S \cap D| = |B \cap T| = 1, |A \cap T| = |S \cap C| = 0, A \cap C = \{y\}$.

Let $S \cap T = \{a, b\}, S \cap D = \{u\}$. It is easy to see that $\Gamma_G(y) = \{x, a, b, z\}, \Gamma_G(z) = \{y, a, b, u\}$.

First, we show that the conclusion of the theorem holds if $az \in E_N(G)$. By Theorem 2.1.1 we consider its corresponding separating group $(az, S_1; A_1, B_1)$ such that $a \in B_1, z \in A_1$. Since $ayza$ is a 3-cycle of G, we have $y \in S_1$, and so $y \in S_1 \cap C$, $a \in B_1 \cap T$. By Theorem 2.1.2 we have $|A_1| = |D| = 2$. If $|A_1 \cap D| = 2$, since $S_1 \cap C \neq \emptyset$, we have $|S_1 \cap T| \leq 2$, and so $\{z\} \cup (S_1 \cap T)$ is a vertex-cut with cardinality less than 4, a contradiction. Therefore, $|A_1 \cap D| = 1$. Since $b \in T$ and $bz \in E(G)$, we have $b \in A_1 \cap T$. Since D is a connected subgraph of G and $|D| = 2$, it is easy to see that $|D \cap S_1| = 1$. Since $zu \in E(G)$, we have $D \cap S_1 = \{u\}$. We claim that $S_1 \cap T = \emptyset$. Otherwise, $|S_1 \cap T| = 1$. Then $|S_1 \cap C| = |B_1 \cap T| = 1$. Obviously, $|(S_1 \cap C) \cup (S_1 \cap T) \cup (B_1 \cap T)| = 3$. Since G is 4-connected, we have $B_1 \cap C = \emptyset$. Therefore, $|C| = |C \cap S_1| = 1$, which contradicts $|C| \geq 2$. Hence, $S_1 \cap T = \emptyset$, and therefore, $|S_1 \cap C| = |B_1 \cap T| = 2$. Here we distinguish the following cases:

(1.) $d(y) = 4, d(a) \geq 5$. Arguments similar to that used in Subcase 2.2.1 can lead to that G contains a W'-framework such that yz is its an inner edge. Then the conclusion holds.

(2.) $d(y) = d(a) = 4$. Arguments similar to that used in Subcase 2.2.3 can lead to that G contains a helm such that yz is its an inner edge. The conclusion holds.

If $bz \in E_N(G)$, by the symmetry of az and bz, similar arguments can be used to get the conclusion. Therefore, we may assume that $az, bz \in E_R(G)$.

Next we consider ay. Assume $ay \in E_N(G)$. By Theorem 2.1.1 we consider its separating group $(ay, S_1; A_1, B_1)$ such that $a \in A_1, y \in B_1$. It is easy to see that $z \in S_1 \cap D$, $y \in B_1 \cap C$ and $a \in A_1 \cap T$. Since $ay, yz \in E_N(G)$, by Theorem 2.1.2 we have $|C| = 2 = |B_1|$, and so $C = \{y, x\}$. By arguments analogous to that used in Case 2, we can get that $|B_1 \cap T| = |S_1 \cap C| = 1, B_1 \cap C = \{y\}$, $|A_1 \cap T| = |D \cap S_1| = 2$. Then $S_1 \cap C = \{x\}$. Since $byzb$ is a 3-cycle of G, it is easy to see that $B_1 \cap T = \{b\}$ and $d(x) = d(b) = d(z) = 4$. Here we distinguish the following cases:

(1.) $d(a) \geq 5$. Argument analogous to that used in Subcase 2.2.1 can lead to that G contains a W'-framework such that xy, yz are its inner edges. Then the conclusion holds.

(2.) $d(a) = 4$. Argument analogous to that used in Subcase 2.2.3 can lead to that G contains a helm such that xy, yz are its inner edges. Then theorem holds.

Thus, we may assume that $ay, by \in E_R(G)$. Then, according to the definition of the l-bi-fan, $(l \geq 1)$, G contains a l-bi-fan such that yz is its an inner edge. This complete the proof. $\square$

Lemma 6.2.1. *Let G be a 4-connected graph with property $(\star)$, and let $P = y_1 y_2 \cdots y_k$ be a path of $[E_N(G)]$ with $k \geq 3$ and take a set D such that $\emptyset \neq D \subset V(G)$. Suppose that $(y_1y_2, U'; X', Y')$ is a separating group of G such that $y_1 \in Y', y_2 \in X'$ and $D \cap Y' \neq \emptyset$. We choose $i \in \{1, 2, \cdots, k\}$ and a separating group $(y_i y_{i+1}, S; A, B)$ satisfying $y_i \in B, y_{i+1} \in A, D \cap B \neq \emptyset$ such that $|A|$ is as small as possible. If $i \leq k-2$, we consider another separating group $(y_{i+1}y_{i+2}, S'; A', B')$ such that $y_{i+1} \in B', y_{i+2} \in A'$, Then one of the following conclusions holds:*

(i) *$A \cap B' = \{y_{i+1}\}, A \cap A' = \{y_{i+2}\}, A \cap S' = \{a\}$, $B' \cap S = \{b\}, S \cap S' = \emptyset, y_i \in B \cap B'$, $|B \cap S'| = |A' \cap S| = 2$, $A' \cap S = \{u, v\}$, where $y_{i+2}u, y_{i+2}v, y_{i+2}a \in E_R(G)$ and $a, b, u, v \in G$.*

(ii) *$A \cap A' = \{y_{i+2}\}, y_{i+1} \in A \cap B', S \cap S' = \emptyset = A' \cap B$, $B \cap S' = \{d\} = D \cap B, D \cap B' = \emptyset, A' \cap S = \{c\}$, $|B' \cap S| = |A \cap S'| = 2, y_i \in B \cap B'$, where $d, c \in G$.*

(iii) *$A \cap A' = \{y_{i+2}\}$, $y_{i+1} \in A \cap B', S \cap S' = \{w\}$, $D \cap B = \{d\} = B \cap S', D \cap B' = \emptyset = B \cap A', A' \cap S = \{c\}$, $|B' \cap S| = |A \cap S'| = 1, y_i \in B \cap B'$, where $d, c, w \in G$.*

(iv) *G contains one of the following structures: l-belt, ($l \geq 1$), helm, W-framework, W'-framework, l-bi-fan, ($l \geq 1$), as its subgraph, such that it intersects P at its some inner edge(s).*

Proof. Let

$$X_1 = (A \cap S') \cup (S \cap S') \cup (B' \cap S)$$

$$X_2 = (A \cap S') \cup (S \cap S') \cup (A' \cap S)$$

$$X_3 = (A' \cap S) \cup (S \cap S') \cup (B \cap S')$$

$$X_4 = (B' \cap S) \cup (S \cap S') \cup (B \cap S')$$

We will distinguish the following cases to complete the proof.

Case 1. $y_i \in B \cap B', y_{i+2} \in A \cap A'$.

Since $B \cap B' \neq \emptyset$, X_4 is a vertex-cut of $G - y_i y_{i+1}$. Since G is 4-connected, we have $|X_4| \geq 3$. By similar arguments we can deduce that $|X_2| \geq 3$. Since $|X_2| + |X_4| = |S| + |S'| = 6$, we have $|X_2| = |X_4|$, and so $|A \cap S'| = |B' \cap S|$, $|A' \cap S| = |B \cap S'|$.

First, we claim that $A' \cap (B \cup S) \neq \emptyset$. Otherwise, $A' \cap (B \cup S) = \emptyset$. Since $|A' \cap S| = 0$, we have $S' \cap B = \emptyset$. Since B is a connected subgraph of G, we have $B = B \cap B'$. Therefore, we have $\emptyset \neq D \cap B = D \cap (B \cap B') \subset D \cap B'$. For the separating group $(y_{i+1}y_{i+2}, S'; A', B')$ of G, we have $y_{i+1} \in B', y_{i+2} \in A', D \cap B' \neq \emptyset$, and $A' \subset A, |A'| < |A|$, which contradicts that $|A|$ is as small as possible, and so $A' \cap (B \cup S) \neq \emptyset$. Since A' is a connected subgraph of G and $A \cap A' \neq \emptyset \neq A' \cap (B \cup S)$, we have $A' \cap S \neq \emptyset \neq B \cap S'$. If $|A' \cap S| = 3$, then $|X_1| = 0$, and so $\{y_i, y_{i+2}\}$ would be a 2-vertex-cut of G, a contradiction. Therefore, $|A' \cap S| = 2$ or $|A' \cap S| = 1$.

Next we distinguish the following subcases.

Subcase 1.1. $|A' \cap S| = |S' \cap B| = 2$.

Let $A' \cap S = \{u, v\}$. Since G is 4-connected and X_1 is a vertex-cut of $G - y_iy_{i+1} - y_{i+1}y_{i+2}$, we have that $|X_1| \geq 2$. Noticing that $|S| = |S'| = 3$, it is easy to see that $|A \cap S'| = |B' \cap S| = 1, |S \cap S'| = 0$. Let $A \cap S' = \{a\}, B' \cap S = \{b\}$. First, we claim that $A \cap B' = \{y_{i+1}\}$. Otherwise, $|A \cap B'| \geq 2$, and so $X_1 \cup \{y_{i+1}\}$ would be a 3-vertex-cut of G, a contradiction. Second, we claim that $A \cap A' = \{y_{i+2}\}$. Otherwise, $|A \cap A'| \geq 2$. Let $A_1 = A \cap A'$, $S_1 = X_2, B_1 = G - y_{i+1}y_{i+2} - S_1 - A_1$. It is easy to see that $D \cap B_1 \neq \emptyset$. Then $(y_{i+1}y_{i+2}, S_1; A_1, B_1)$ is a separating group of G such that $y_{i+1} \in B_1, y_{i+2} \in A_1$ and $D \cap B_1 \neq \emptyset$. However, $|A_1| < |A|$, which contradicts that $|A|$ is as small as possible. Therefore, $A \cap A' = \{y_{i+2}\}$. Obviously, (ab, S_1) is a separating pair of G such that $S_1 = \{y_{i+1}, u, v\}$, and so $ab \in E_N(G)$. We claim that $y_{i+2}u, y_{i+2}v \in E_R(G)$. Otherwise, $\{y_{i+2}u, y_{i+2}v\} \cap E_N(G) \neq \emptyset$. From Lemma 5.1.2 we have that au, av can not belong to $E(G)$ simultaneously. Without loss of generality, we may assume that $au \notin E(G)$. Let $A_1 = A - \{y_{i+2}\}, S_1 = \{y_{i+2}\} \cup (S - \{u\}), B_1 = G - y_iy_{i+1} - S_1 - A_1$, then $(y_iy_{i+1}, S_1; A_1, B_1)$ is a separating group of G such that $D \cap B_1 \neq \emptyset$. However, $|A_1| < |A|$, which contradicts that $|A|$ is as small as possible. Therefore, $y_{i+2}u, y_{i+2}v \in E_R(G)$. We claim that $ay_{i+2} \in E_R(G)$. Otherwise, $ay_{i+2} \in E_N(G)$, and we take its separating group $(ay_{i+2}, T'; C', D')$ such that $a \in C', y_{i+2} \in D'$. Since $ay_{i+1}y_{i+2}a$ is a 3-cycle of G, we have that $y_{i+1} \in T'$. Since $y_{i+1}y_{i+2} \in E_N(G)$, from Theorem 2.1.2 we have that $|D'| = 2$. Let $D' = \{y_{i+2}, v_1\}$, then $v_1y_{i+1}y_{i+2}v_1$ is a 3-cycle of G and $v_1 \neq a$. Obviously, it is impossible to hold in G, and hence, $ay_{i+2} \in E_R(G)$. Then the conclusion (i) holds.

Subcase 1.2. $|A' \cap S| = |B \cap S'| = 1$.

Let $A' \cap S = \{c\}, B \cap S' = \{d\}$. Then we will discuss the following subcases.

Subcase 1.2.1. $|S \cap S'| = 0, |B' \cap S| = |A \cap S'| = 2$.

It is easy to see that $|X_3| = 2$. Since G is 4-connected, we have $A' \cap B = \emptyset$ and $|X_2| = 3$. We claim $A \cap A' = \{y_{i+2}\}$. Otherwise, $|A \cap A'| \geq 2$. Let $A_1 = A \cap A', S_1 = X_2, B_1 = G - y_{i+1}y_{i+2} - S_1 - A_1$. Then $(y_{i+1}y_{i+2}, S_1; A_1, B_1)$ is a separating group. Obviously, $D \cap B_1 \neq \emptyset$ and $|A_1| < |A|$, which contradicts that $|A|$ is as small as possible. Therefore, $A \cap A' = \{y_{i+2}\}$, and so $A' = \{y_{i+2}, c\}, |A'| = 2 < |A|$. By the minimum property of $|A|$, we have $B' \cap D = \emptyset$. Therefore, $B \cap D = B \cap S' = \{d\}$ and $|B \cap D| = 1$. Then conclusion (ii) holds.

Subcase 1.2.2. $|S \cap S'| = 1, |B' \cap S| = |A \cap S'| = 1$.

Let $A' \cap S = \{c\}$, $S \cap S' = \{w\}, B \cap S' = \{d\}$. Since $|X_3| = 3 < 4$, we have $B \cap A' = \emptyset$. Arguments similar to that used in Subcase 1.2.1 can lead to that $A \cap A' = \{y_{i+2}\}, y_{i+1} \in A \cap B'$. Since $|A'| = 2 < |A|$, by arguments similar to that used in Subcase 1.2.1, we have that $B' \cap D = \emptyset$, and so $D \cap B = B \cap S' = \{d\}$. Then conclusion (iii) holds.

Subcase 1.2.3. $|S \cap S'| = 2, |B' \cap S| = |A \cap S'| = 0$,

Let $S \cap S' = \{a, b\}$. We claim that $A \cap B' = \{y_{i+1}\}$. Otherwise, $|A \cap B'| \geq 2$. Then $\{y_{i+1}, a, b\}$ is a 3-vertex-cut of G, which contradicts that G is 4-connected. It is easy to see that $|X_2| = 3$. Arguments similar to that used in Subcase 1.2.1 can lead to that $A \cap A' = \{y_{i+2}\}$. By Corollary 2.1.3 we have that $\{ay_{i+1}, ay_{i+2}\} \cap E_R(G) \neq \emptyset$, $\{by_{i+1}, by_{i+2}\} \cap E_R(G) \neq \emptyset$. Next we distinguish the following cases.

(1.) $ay_{i+2} \in E_N(G)$. Then $A' \cap B = \emptyset$ and we consider the corresponding separating group $(ay_{i+2}, S_1; A_1, B_1)$ such that $y_{i+2} \in A_1, a \in B_1$. Since $ay_{i+1}y_{i+2}a$ is a 3-cycle of G, we have $y_{i+1} \in S_1$, and so $y_{i+1} \in S_1 \cap B'$. Since $a \in S'$, we have $a \in S' \cap B_1$. Obviously, $d(y_{i+1}) = d(y_{i+2}) = 4$. By arguments analogous to that used in Subcase 2.2 of Theorem 6.2.3, we can get that $y_{i+1}y_{i+2}$ is an inner edge of a W'-framework or a helm, and so conclusion (iv) holds. For $by_{i+2} \in E_N(G)$, we may apply similar arguments to get conclusion (iv) holds.

Hence, we may assume that $ay_{i+2}, by_{i+2} \in E_R(G)$.

(2.) $ay_{i+1} \in E_N(G)$. We consider the corresponding separating group $(ay_{i+1}, S_1; A_1, B_1)$ such that $y_{i+1} \in A_1, a \in B_1$. Then $y_{i+1} \in A_1 \cap B', a \in B_1 \cap S'$. Since $ay_{i+1}y_{i+2}a$ is a 3-cycle of G, we have $y_{i+2} \in S_1$, and so $y_{i+2} \in A' \cap S_1$. Since $ay_{i+2} \in E(G)$ and $d(y_{i+2}) = 4$, by arguments analogous to that used in Subcase 2.2 of Theorem 6.2.3 we can get that $y_{i+1}y_{i+2}$ is an inner edge of a W'-framework or a helm. Therefore, conclusion (iv) holds. For $by_{i+1} \in E_N(G)$, we may apply similar arguments to get conclusion (iv) holds.

Based on the above arguments, we may assume that $ay_{i+1}, by_{i+1}, ay_{i+2}, by_{i+2} \in E_R(G)$, and so G contains a l-bi-fan such that $y_{i+1}y_{i+2}$ is its an inner edge. Therefore, conclusion (iv) holds.

Case 2. $y_{i+2} \in A \cap A', y_i \in B \cap S'$.

Since $y_iy_{i+1} \in E_N(G)$, by Theorem 2.1.2 we have that $|B'| = 2$. Since B' is a connected subgraph of G, we have $B \cap B' = \emptyset$. Because G is 4-connected and X_1 is a vertex-cut of $G - y_iy_{i+1} - y_{i+1}y_{i+2}$, we have $|X_1| \geq 2$. Similar arguments can lead to that $|X_2| \geq 3$. We claim that $A \cap B' = \{y_{i+1}\}$. If not, i.e., $|A \cap B'| = 2$, by $B \cap S' \neq \emptyset$ and $|S'| = 3$ we have $|X_1| \leq 2$, and so $X_1 \cup \{y_{i+1}\}$ is a vertex-cut of G with cardinality less than 4, which contradicts that G is 4-connected. Therefore, $|A \cap B'| = |B' \cap S| = 1$. If $|B \cap S'| = 1$, then $|X_3| = 3$, and so $A' \cap B = \emptyset$. Then we have $|B| = |B \cap S'| = 1$, which contradicts $|B| \geq 2$. Hence $|B \cap S'| \geq 2$. If $|B \cap S'| = 3$, then we have $A \cap S' = \emptyset = S \cap S'$, and so $|X_1| = 1$, which contradicts $|X_1| \geq 2$. Therefore, $|B \cap S'| = 2$ and $|S \cap S'| \leq 1$. If $|S \cap S'| = 1$, then $A \cap S' = \emptyset$ and $|A' \cap S| = 1$, and hence $|X_2| = 2$, which contradicts $|X_2| \geq 3$. Then we can get that $S \cap S' = \emptyset$ and $|A \cap S'| = 1$. By $|S| = 3$ we know that $|A' \cap S| = 2, |X_2| = 3$. We claim that $A \cap A' = \{y_{i+2}\}$. If not, i.e., $|A \cap A'| \geq 2$, then we let $A_1 = A \cap A', S_1 = X_2, B_1 = G - y_{i+1}y_{i+2} - S_1 - A_1$. Then $(y_{i+1}y_{i+2}, S_1; A_1, B_1)$ is a separating group of G. It is easy to see that $B_1 \cap D \neq \emptyset$. However, now we have $|A_1| < |A|$, which contradicts that $|A|$ is as small as possible. Therefore, $A \cap A' = \{y_{i+2}\}$.

Let $A \cap S' = \{a\}, B' \cap S = \{b\}$. Next we show that $by_i, by_{i+1}, ay_{i+1} \in E_R(G)$ by contradiction.

(1.) If $by_i \in E_N(G)$. We consider its corresponding separating group $(by_i, T; C, K)$ of G such that $b \in C, y_i \in K$. Since $by_iy_{i+1}b$ is a 3-cycle of G, we have $y_{i+1} \in T$. Since $y_iy_{i+1} \in E_N(G)$, by Theorem 2.1.2 we can get $|K| = 2$, say $K = \{y_i, v_1\}$. Then $v_1y_{i+1}y_iv_1$ is a 3-cycle of G and $v_1 \neq b$, which is impossible in G. Hence $by_i \in E_R(G)$.

(2.) If $by_{i+1} \in E_N(G)$. Similarly we consider its corresponding separating group $(by_{i+1}, T; C, K)$ of G such that $b \in C, y_{i+1} \in K$. It is easy to see that $\{a, y_i\} \subset T$. Since $y_iy_{i+1} \in E_N(G)$, by Theorem 2.1.2 we have $|K| = 2$, say $K = \{y_{i+1}, v_1\}$. Then $v_1 \in \Gamma_G(y_i) \cap \Gamma_G(y_{i+1}) \cap \Gamma_G(a)$, which is impossible in G, and so $by_{i+1} \in E_R(G)$.

(3.) If $ay_{i+1} \in E_N(G)$. Again similarly we consider its corresponding separating group $(ay_{i+1}, T; C, K)$ such that $a \in C, y_{i+1} \in K$. Since $ay_{i+1}y_{i+2}a$ is a 3-cycle of G, we have $y_{i+2} \in T$. Since $y_{i+1}y_{i+2} \in E_N(G)$, by Theorem 2.1.2 we have $|K| = 2$. Let $K = \{y_{i+1}, v_1\}$, then $y_{i+1}v_1y_{i+2}y_{i+1}$ is a 3-cycle of G, and $v_1 \neq a$, which is impossible in G, and so $ay_{i+1} \in E_R(G)$.

Let $A_1 = \{a, y_{i+2}\}, S_1 = S \cap A' \cup \{y_{i+1}\}$ and $B_1 = G - ab - S_1 - A_1$. Then $(ab, S_1; A_1, B_1)$ is a separating group of G, and so $ab \in E_N(G)$.

Noticing that $d(b) = d(y_{i+1}) = 4$, from the definition of an l-belt we know that G contains an l-belt with y_iy_{i+1} as an inner edge. Therefore, conclusion (iv) holds.

Case 3. $y_i \in B \cap S', y_{i+2} \in A' \cap S$.

By Theorem 2.1.2 we have $|A| = 2, |B'| = 2$. Since A and B' are connected subgraphs of G, it follows $A \cap A' = \emptyset = B \cap B'$. If $|A \cap B'| = 2$, then $B' \cap S = \emptyset = A \cap S'$. Since $B \cap S' \neq \emptyset \neq A' \cap S$, noticing that $|S| = |S'| = 3$, we have $|S \cap S'| \leq 2$, and so $\{y_{i+1}\} \cup (S \cap S')$ is a vertex-cut of G with cardinality less than 4, which contradicts the fact that G is 4-connected. Therefore, $A \cap B' = \{y_{i+1}\}$, and so $|B' \cap S| = |A \cap S'| = 1$. If $|A' \cap S| = 1$, then $A' \cap B \neq \emptyset$. Then X_3 is a vertex-cut of G, and so $|X_3| \geq 4$. Then, $1 = |A' \cap S| > |A \cap S'| = 1$, a contradiction. Hence, $|A' \cap S| = 2$, and so $S \cap S' = \emptyset, |B \cap S'| = 2$. By arguments similar to those used in Case 2 of Theorem 6.2.3, we know that conclusion (iv) of the lemma holds.

Case 4. $y_i \in B \cap B', y_{i+2} \in A' \cap S$.

Arguments analogous to that used in Case 1 of Theorem 6.2.3 can show that G contains an l-belt with $y_{i+1}y_{i+2}$ as an inner edge. Therefore, conclusion (iv) of the lemma holds. This complete the proof. $\square$

Theorem 6.2.4 *Let G be a 4-connected graph with property $(\star)$. Suppose that H is a helm of G such that H is defined as in Definition 1.2.1. Let $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $P = y_1y_2 \cdots y_h$ a path in $[E_N(G)]$ with $h \geq 2$ such that $a \notin V(P)$ and $\{y_1, y_h\} \subset \{x_1, x_2, x_3, x_4\}$. Then, G contains one of the following structures H_1 as subgraph: l-belt, l-bi-fan, $(l \geq 1)$, W-framework, W'-framework or helm, such that at least one inner edge of H_1 belongs to $E(P \cup H)$, and H and H_1 do not have any common inner edge.*

Proof. Without loss of generality, we assume that $y_1 = x_1$. Then it is easy to see that $y_2 = v_1$. Let $k = h + 1, y_k = a$, then $P' = y_1y_2 \cdots y_k$ is also a path of $[E_N(G)]$ where $k \geq 3$. Let $D = \{a\}$. We consider the separating group $(x_1v_1, S_1; A_1, B_1)$ such that $S_1 = \{x_2, x_3, x_4\}, B_1 = \{x_1, a\}, A_1 = G - x_1v_1 - S_1 - B_1$. Obviously, $D \cap B_1 \neq \emptyset$.

We consider the separating group $(y_iy_{i+1}, S; A, B)$ of G, where $i = 1, 2, \cdots, k-1$, such that $y_i \in B, y_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is as small as possible. We claim that $i + 1 \leq k - 1$ holds. Otherwise, $y_{i+1} = y_k$, i.e., $y_{i+1} = a$. Then, $a \in A \cup S$, which contradicts $D \cap B \neq \emptyset$. Therefore, $i + 1 \leq k - 1$.

We consider another separating group $(y_{i+1}y_{i+2}, S'; A', B')$ such that $y_{i+1} \in B', y_{i+2} \in A'$, and $|A'|$ is as small as possible. We know that one of the four conclusions of Lemma 6.2.1 holds. Now we discuss them as follows.

(1.) Conclusion (i) of Lemma 6.2.1 holds. It is easy to see that $P' + ax_1$ is a cycle of $[E_N(G)]$. Then each vertex of P is incident with at least two unremovable edges of G. However, from conclusion (i) we have that $d(y_{i+2}) = 4$ and y_{i+2} is incident with three removable edges of G. Therefore, conclusion (i) can not hold.

(2.) Conclusion (ii) of Lemma 6.2.1 holds. Then $B \cap S' = \{d\} = \{a\} = D \cap B$, $c \in \{x_1, x_2, x_3, x_4\}$, and $ac(= dc)$ is not in any 3-cycle of G. However, from the definition of the helm, we know that $ac(= ax_j)$ for each $j = 1, 2, 3, 4$ is in two 3-cycles of G, a contradiction.

(3.) Conclusion (iii) of Lemma 6.2.1 holds. Then $\{d\} = B \cap S' = \{a\} = D \cap B$. Since $ac \in E(G)$, we have $c \in \{x_1, x_2, x_3, x_4\}$. Then ac belongs to two 3-cycles of G. However, this is impossible in G. Therefore, conclusion (iii) cannot hold.

(4.) If conclusion (iv) of Lemma 6.2.1 holds. Then the theorem holds. This completes the proof. □

Theorem 6.2.5 *Let G be a 4-connected graph with property $(\star)$ and let L_1 be a maximal 1-belt of G as defined in Definition 1.2.3 such that $V(L_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Suppose that $P = l_1 l_2 \cdots l_h$ is a path of $[E_N(G)]$ such that $\{l_1, l_h\} \subset \{x_1, x_3, y_1,$*
$y_3\}$ and $\{x_2, y_2\} \cap V(P) = \emptyset$. Then G contains one of the following structures L' as subgraph: l-belt, $(l \geq 1)$, helm, W-framework, W'-framework or l-bi-fan, $(l \geq 1)$, such that at least one inner edge of L' belongs to $E(P \cup L_1)$.

Proof. We consider the following cases.

Case 1. $l_h = y_3$.

By letting $k = h+1, l_k = y_2$, then $P' = l_1 l_2 \cdots l_k$ is also a path of $[E_N(G)]$. Let $D = \{x_2, y_2\}$, and consider a separating group $(l_1 l_2, S_1; A_1, B_1)$ of G such that $l_1 \in B_1, l_2 \in A_1$. Next we will show that $B_1 \cap D \neq \emptyset$. We distinguish the following subcases.

Subcase 1.1. $l_1 = x_1$.

We claim that $x_2 \in B_1$. Otherwise, $x_2 \in S_1$. Since $x_1 x_2 \in E_N(G)$, by Theorem 2.1.2 we have $|B_1| = 2$. Let $B_1 = \{l_1, v_1\}$, then $v_1 \in \Gamma_G(x_1) \cap \Gamma_G(x_2)$. If $v_1 = y_1$, then $\Gamma_G(y_1) = \{x_1, x_2, y_2, w\}$, where $w \in V(G)$, which contradicts the assumption that L_1 is a maximal 1-belt. If $v_1 = x_3$, then $\Gamma_G(x_3) = \{x_2, y_2, x_1, w\}$. It is easy to see that $(x_2 y_1, T)$ is a separating pair of G such that $T = \{w, y_2, x_1\}$, and so $x_2 y_1 \in E_N(G)$, which contradicts the definition of the l-belt. Therefore, $x_2 \in B_1$ holds, i.e., $D \cap B_1 \neq \emptyset$.

Subcase 1.2. $l_1 = y_1$.

Then if $y_2 \in S_1$, since $y_1 y_2 \in E_N(G)$, by Theorem 2.1.2 we have $|B_1| = 2$. It is easy to see that $B_1 = \{y_1, x_2\}$, and so $D \cap B_1 \neq \emptyset$. If $y_2 \in B_1$, then $D \cap B_1 \neq \emptyset$.

Subcase 1.3. $l_1 = x_3$.

We claim that $D \cap B_1 \neq \emptyset$. Otherwise, $D \cap B_1 = \emptyset$. Since $x_3 y_2, x_3 x_2 \in E(G)$, we have $x_2, y_2 \in S_1$. Since $x_2 x_3 \in E_N(G)$, by Theorem 2.1.2 we have $|B_1| = 2$. Let $B_1 = \{x_3, v_1\}$, then it is easy to see that $v_1 \in \Gamma_G(x_2) \cap \Gamma_G(y_2) \cap \Gamma_G(x_3)$. Then $v_1 = y_1$ holds, i.e., $y_1 x_3 \in E(G)$. Since $x_2 x_3 \in E_N(G)$, we consider the separating group $(x_2 x_3, T_1; C_1, D_1)$ such that $x_2 \in C_1, x_3 \in D_1$. Then $y_1, y_2 \in T_1$. By Theorem 2.1.4, we have $y_1 y_2 \in E_R(G)$, which contradicts the definition of the l-belt. Therefore, $D \cap B_1 \neq \emptyset$.

We consider the separating group $(l_i l_{i+1}, S; A, B)$ of G such that $l_i \in B, l_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is as small as possible. We claim that $i+1 \leq k-1$. Otherwise, $i+1 = k$ holds. Then $l_k = y_2$. From $x_2 y_2 \in E(G)$

we have that $\{x_2, y_2\} \subset A \cup S$, which contradicts $D \cap B \neq \emptyset$. Therefore, $i+1 \leq k-1$ holds.

Case 2. $l_h = x_3$.

We consider the separating group $(l_1l_2, S_1; A_1, B_1)$ of G such that $l_1 \in B_1, l_2 \in A_1$. Let $D = \{x_2, y_2\}$. Similarly, next we need to show that $D \cap B_1 \neq \emptyset$.

If $l_1 = y_1$. From $y_1y_2 \in E(G)$ we have that $y_2 \in B_1 \cup S_1$. If $y_2 \in S_1$, since $y_1y_2 \in E_N(G)$, by Theorem 2.1.2 we obtain $|B_1| = 2$. Let $B_1 = \{y_1, v_1\}$. Then $y_1y_2v_1y_1$ is a 3-cycle of G. It is easy to see that $v_1 = x_2$. Then $D \cap B_1 \neq \emptyset$.

By the symmetry of the maximal 1-belt, for the other cases we may apply similar arguments to prove $D \cap B_1 \neq \emptyset$.

Now we consider the separating group $(l_il_{i+1}, S; A, B)$ such that $l_i \in B, l_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is small as possible, where $i = 1, 2 \cdots, h-1$. We claim that $i+1 \leq h-1$. Otherwise, $l_h = x_3 \in A$. Since $x_2x_3, y_2x_3 \in E(G)$, we have that $x_2, y_2 \in A \cup S$, which contradicts $D \cap B \neq \emptyset$.

We consider the separating group $(l_{i+1}l_{i+2}, S'; A', B')$ of G such that $l_{i+1} \in B', l_{i+2} \in A'$ and $|A'|$ is as small as possible. Then one of the four conclusions of Lemma 6.2.1 holds. Here we discuss them as follows.

(1.) It is easy to see that each vertex of P is incident with at least two unremovable edges, and so conclusion (i) of Lemma 6.2.1 cannot hold.

(2.) If conclusion (ii) of Lemma 6.2.1 holds, then we have that $B \cap S' = D \cap B = \{d\} \subset \{x_2, y_2\}$. By the symmetry between x_2 and y_2, without loss of generality, we may assume that $d = x_2$. For $d = y_2$, we may apply similar arguments.

By Lemma 6.2.1, we know that $A \cap A' = \{l_{i+2}\}, l_{i+1} \in A \cap B'$. Let $A \cap S' = \{v_1, v_2\}$. If $v_1 l_{i+2} \in E_N(G)$, we consider the corresponding separating group $(v_1 l_{i+2}, T; C, K)$ such that $v_1 \in C, l_{i+2} \in K$, and so $v_1 \in S' \cap C$.

(2.1.) If $l_{i+1} \in B' \cap K$. By the arguments analogous to that used in Case 1 of Theorem 6.2.3, we can get that $|A'| = 2, |K \cap A'| = |A' \cap T| = 1, |C \cap S'| = 2, |S' \cap K| = 1$. Let $K \cap S' = \{b\}, A' \cap T = \{a\}, S' \cap C = \{v_1, w\}$. Then by arguments analogous to that used in Case 1 of Theorem 6.2.3, we have that $al_{i+2}, av_1 \in E_R(G), bl_{i+2} \in E_R(G)$, $ab \in E_N(G), d(a) = d(l_{i+2}) = 4$. It is easy to see that the l-belt is a subgraph of G, where $l \geq 1$, and $\Gamma_G(l_{i+2}) = \{l_{i+1}, v_1, a, b\}$. We claim that l_{i+2} is not an end-vertex of P. Otherwise, we have $l_{i+2} \in \{x_1, x_3, y_1, y_3\}$. Since $B \cap S' = \{x_2\}$, and $x_1, x_3, y_1 \in \Gamma_G(x_2)$, then this is true only if $l_{i+2} = y_3$ holds. Let $A' \cap S = \{k\}$. Noticing that (kx_2, T') is the separating pair of G such that $T' = \{l_{i+1}\} \cup (S' - \{x_2\})$, we have $k \in \{x_3, x_1\}$. If $k = x_3$, then $x_3 y_3 \in E(G)$ and $d(x_3) = 4$ must hold, which contradicts the definition of the maximal 1-belt. If $k = x_1$, noticing that $y_2 \notin V(P)$, then $l_{i+1} \neq y_2$, and so we have that $x_1 y_2 \in E(G)$, a contradiction. Therefore, l_{i+2} is not an end-vertex of P. Since $al_{i+2}, bl_{i+2} \in E_R(G)$, we have that $l_{i+2} v_1 \in E(P)$ and $l_{i+2} v_1$ is an inner edge of the l-belt. Hence, the theorem holds.

(2.2.) If $l_{i+1} \in B' \cap T$. Then by arguments analogous to that used in Case 2 of Theorem 6.2.3, we have that $l_{i+1} l_{i+2}$ is an inner edge of one of the following subgraphs of G: helm, W'-framework, W-framework or l-belt. Therefore, the theorem holds.

So, next we may assume that $v_1 l_{i+2} \in E_R(G)$. For the case $v_2 l_{i+2} \in E_N(G)$, we may apply similar arguments as the case of $v_1 l_{i+2} \in E_N(G)$. So, next we may assume that $v_2 l_{i+2} \in E_R(G)$. Let $A' \cap S = \{c\}$. Since P is a path of $[E_N(G)]$, and l_{i+2} is not an end-vertex of P, we have $l_{i+2} c \in E_N(G) \cap E(P)$. If $cv_1 \in E_N(G)$, we consider the corresponding separating group $(cv_1, T'; C', D')$ of G such that $v_1 \in C', c \in D'$. Obviously, $l_{i+2} \in T'$. Since $cl_{i+2} \in E_N(G)$, by Theorem 2.1.2 we have $|D'| = 2$, and so $D' = \{c, v_2\}$. Then, $|\Gamma_G(c) \cap \Gamma_G(v_2)| \geq 2$. Noticing that $v_1 \in C_1$, obviously this is impossible to hold in G. So,

$cv_1 \in E_R(G)$. By analogous arguments, we have $cv_2 \in E_R(G)$. It is easy to see that cl_{i+2} is an inner edge of an l-bi-fan, and so the theorem holds.

(3.) If conclusion (iii) of Lemma 6.2.1 holds. Then we have $B \cap S' = D \cap B = \{d\} \subset \{x_2, y_2\}$. By the symmetry of x_2 and y_2, we may assume that $d = y_2$. Let $A \cap S' = \{v_1\}$, $S \cap S' = \{w\}, A' \cap S = \{c\}$. Then $\Gamma_G(c) = \{l_{i+2}, v_1, w, y_2\}$. Since $cw \in E([S])$, by Theorem 2.1.4 we obtain $cw \in E_R(G)$. By analogous arguments as used in (2.1.) we can show that l_{i+2} is not an end-vertex of P.

(3.1.) If $l_{i+2}v_1 \in E_N(G)$. We consider the corresponding separating group $(l_{i+2}v_1, T; C, K)$ such that $l_{i+2} \in K, v_1 \in C$. Then $l_{i+2} \in A' \cap K, v_1 \in C \cap S', l_{i+1} \in B'$. We claim that $l_{i+1} \notin B' \cap K$. Otherwise, $l_{i+1} \in B' \cap K, A' = \{l_{i+2}, c\}$. By arguments analogous to that used in Case 1 of Theorem 6.2.3, we can get that $A' \cap K = \{l_{i+2}\}, A' \cap T = \{c\}, T \cap S' = \emptyset, |T \cap B'| = |C \cap S'| = 2, |K \cap S'| = 1$. Since $wl_{i+2} \in E(G)$, we have $w \in K \cap S'$. Let $A_2 = (K \cap B') \cup \{w\}, S_2 = (T \cap B') \cup \{l_{i+2}\}, B_2 = G - cw - S_2 - A_2$. Then $(cw, S_2; A_2, B_2)$ is a separating group of G. So, $cw \in E_N(G)$, an contradiction to $cw \in E_R(G)$. Hence, $l_{i+1} \notin B' \cap K$, and so $l_{i+1} \in B' \cap T$. By arguments analogous to that used in Case 2 of Theorem 6.2.3, we have $|A'| = |K| = 2$ and $|K \cap S'| = |A' \cap T| = 1$. Noticing that $c \in A', w \in S', \Gamma_G(l_{i+2}) = \{l_{i+1}, c, w, v_1\}$, it is easy to see that $K \cap S' = \{w\}, A' \cap T = \{c\}$. By arguments analogous to that used in Case 2 of Theorem 6.2.3, and noticing that $cw \in E_R(G)$, we see that $l_{i+1}l_{i+2}$ is an inner edge of one of the following subgraphs of G: W'-framework, W-framework or helm. Therefore, the theorem holds.

So, next we may assume that $l_{i+2}v_1 \in E_R(G)$.

(3.2.) If $wl_{i+2} \in E_N(G)$. We consider the corresponding separating group $(wl_{i+2}, T'; C', D')$ of G such that $w \in C', l_{i+2} \in D'$. Then $w \in S' \cap C'$.

(3.2.1.) If $l_{i+1} \in B' \cap D'$. By arguments analogous to that used in Case 1 of Theorem 6.2.3, we know that wl_{i+2} is an inner edge of an l-belt, where $l \geq 1$, and $cl_{i+2} \in E_R(G)$. Since l_{i+2} is incident with only two unremovable edges $l_{i+1}l_{i+2}, wl_{i+2}$, and l_{i+2} is not an end-vertex of P, we have $wl_{i+2} \in E(P)$. Hence, the theorem holds.

(3.2.2.) If $l_{i+1} \in B' \cap T'$. Then by an argument analogous to that used in Case 2 of Theorem 6.2.3, we know that $l_{i+1}l_{i+2}$ is an inner edge of one of the following subgraphs of G: l-belt, W-framework, W'-framework or helm, and so the theorem holds.

Therefore, next we may assume that $wl_{i+2} \in E_R(G)$.

Since $E(P) \subset E_N(G)$, we have $cl_{i+2} \in E_N(G)$. If $cv_1 \in E_N(G)$, we consider the corresponding separating group $(cv_1, T'; C', D')$ such that $v_1 \in C', c \in D'$. Obviously, $l_{i+2} \in T'$. Since $cl_{i+2} \in E_N(G)$, by Theorem 2.1.2 we obtain $|D'| = 2$. Let $D' = \{u, c\}$, then $cul_{i+2}c$ is a 3-cycle of G, and so this is true only if $u = w$ holds. Since $cy_2(= cd) \in E(G)$, we have $y_2 \in T'$, and so $wy_2 \in E(G)$. We consider the separating group $(cl_{i+2}, T_1; C_1, D_1)$ such that $c \in C_1, l_{i+2} \in D_1$. Since $cv_1l_{i+2}c$ is a 3-cycle of G, we conclude $v_1 \in T_1$. Then we have $l_{i+2} \in D_1 \cap T', v_1 \in C' \cap T_1, c \in D' \cap C_1$. By arguments analogous to that used in Case 2 of Theorem 6.2.3, and by noticing that $d(l_{i+2}) = 4$, and $v_1l_{i+2} \in E(G)$, we can get that cl_{i+2} is an inner edge of one of the following subgraphs of G: W'-framework or helm. Therefore, the theorem holds.

So, next we may assume $cv_1 \in E_R(G)$. It is easy to see that G contains an l-bi-fan such that cl_{i+2} is an inner edge, where $l \geq 1$. Analogous arguments can lead to $cl_{i+2} \in E(P)$. So, the theorem holds.

(4.) If conclusion (iv) of Lemma 6.2.1 holds. Then the Theorem holds. This completes the proof. $\square$

Corollary 6.2.1 *Let G be a 4-connected graph with property $(\star)$ and let L'_1 be a maximal 1-co-belt of G defined as in Definition 1.2.4 with $V(L'_1) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}$. Suppose that $P = l_1l_2 \cdots l_h$ is a path of $[E_N(G)]$ such that $\{x_2, x_3, y_2\} \cap V(P) = \emptyset$ and $\{l_1, l_h\} \subset \{x_1, x_4, y_1, y_3\}$. Then, G contains* *one of the following structures as subgraph: l-belt, $(l \geq 1)$, W-framework, W'-framework, helm or l-bi-fan, $(l \geq 1)$, such that it has some inner edge(s) belonging to $E(P)$.*

Proof. We distinguish the following cases to complete the proof.

Case 1. $l_h = x_4$. By letting $k = h + 1, l_k = x_3$, then $P' = l_1 l_2 \cdots l_k$ is also a path of $[E_N(G)]$. Let $D = \{x_2, x_3, y_2\}$, and consider a separating group $(l_1 l_2, S_1; A_1, B_1)$ of G such that $l_1 \in B_1, l_2 \in A_1$. Next we show that $B_1 \cap D \neq \emptyset$. We distinguish the following subcases.

Subcase 1.1. $l_1 = x_1$.

We claim that $x_2 \in B_1$. Otherwise, $x_2 \in S_1$. Since $x_1 x_2 \in E_N(G)$, by Theorem 2.1.2 we have $|B_1| = 2$. Let $B_1 = \{l_1, v_1\}$, then $v_1 \in \Gamma_G(x_1) \cap \Gamma_G(x_2)$. If $v_1 = y_1$, then $\Gamma_G(y_1) = \{x_1, x_2, y_2, w\}$, where $w \in V(G)$, which contradicts the assumption that L'_1 is a maximal 1-co-belt. Obviously, $v_1 \notin \{x_3, y_2\}$, and therefore $x_2 \in B_1$ holds, i.e., $D \cap B_1 \neq \emptyset$.

Subcase 1.2. $l_1 = y_1$.

Then if $y_2 \in S_1$, since $y_1 y_2 \in E_N(G)$, by Theorem 2.1.2 we have that $|B_1| = 2$. It is easy to see that $B_1 = \{y_1, x_2\}$, and so $D \cap B_1 \neq \emptyset$. If $y_2 \in B_1$, then $D \cap B_1 \neq \emptyset$.

Subcase 1.3. $l_1 = y_3$.

We claim that $D \cap B_1 \neq \emptyset$. Otherwise, $D \cap B_1 = \emptyset$. Since $x_3 y_3, y_2 y_3 \in E(G)$, we have $x_3, y_2 \in S_1$. Since $y_2 y_3 \in E_N(G)$, by Theorem 2.1.2 we have $|B_1| = 2$. Let $B_1 = \{y_3, v_1\}$, then it is easy to see that $v_1 \in \Gamma_G(y_2) \cap \Gamma_G(y_3) \cap \Gamma_G(x_3)$, which is impossible to hold in G. Therefore, $D \cap B_1 \neq \emptyset$.

We consider the separating group $(l_i l_{i+1}, S; A, B)$ of G such that $l_i \in B, l_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is as small as possible. We claim that $i + 1 \leq k - 1$. Otherwise, $i + 1 = k$. Then $l_k = x_3$. Since $x_2 x_3, y_2 x_3 \in E(G)$, we have $\{x_2, x_3, y_2\} \subset A \cup S$, which contradicts $D \cap B \neq \emptyset$. Therefore, $i+1 \leq k-1$ holds.

Case 2. $l_h = y_3$.

By letting $k = h+1, l_k = y_2$, then $P' = l_1 l_2 \cdots l_k$ is also a path of $[E_N(G)]$. Let $D = \{x_2, x_3, y_2\}$. We consider the separating group $(l_1 l_2, S_1; A_1, B_1)$ of G such that $l_1 \in B_1, l_2 \in A_1$. Similarly, next we need to show that $D \cap B_1 \neq \emptyset$.

If $l_1 = y_1$. Since $y_1 y_2, y_1 x_2 \in E(G)$, we have $x_2, y_2 \in B_1 \cup S_1$. If $x_2, y_2 \in S_1$, since $y_1 y_2 \in E_N(G)$, by Theorem 2.1.2 we obtain $|B_1| = 2$. Let $B_1 = \{y_1, v_1\}$. Then $v_1 = \Gamma_G(y_1) \cap \Gamma_G(y_2) \cap \Gamma_G(x_2)$, which is impossible to hold in G. Then $D \cap B_1 \neq \emptyset$.

By the symmetry of the maximal 1-co-belt, for the other cases we may apply similar arguments.

We consider the separating group $(l_i l_{i+1}, S; A, B)$ such that $l_i \in B, l_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is small as possible, where $i = 1, 2 \cdots, k-1$. We claim that $i+1 \leq k-1$. Otherwise, $l_k = y_2 \in A$. Since $x_2 y_2, y_2 x_3 \in E(G)$, we have $x_2, x_3, y_2 \in A \cup S$, which contradicts $D \cap B \neq \emptyset$.

We consider the separating group $(l_{i+1} l_{i+2}, S'; A', B')$ of G such that $l_{i+1} \in B', l_{i+2} \in A'$ and $|A'|$ is as small as possible. By Lemma 6.2.1 we have that one of the four conclusions of Lemma 6.2.1 holds. Here we will discuss them as follows.

(1.) It is easy to see that each vertex of P is incident with at least two unremovable edges, and so conclusion (i) of Lemma 6.2.1 cannot hold.

(2.) If conclusion (ii) of Lemma 6.2.1 holds. Then we have $B \cap S' = D \cap B = \{d\} \subset \{x_2, x_3, y_2\}$.

First, we claim that l_{i+2} is not the end-vertex of P. Otherwise, we assume that $l_{i+2} \in \{x_1, x_4, y_1, y_3\}$ holds. Let $A' \cap S = \{k\}$. Noticing that (kd, T') is a separating pair of G such that $T' = \{l_{i+1}\} \cup (S' - \{d\})$, so $kd \in E_N(G)$. If $d = x_2$, since $x_1x_2, x_2y_1 \in E(G)$, we have that $l_{i+2} \in \{y_3, x_4\}$: (1.) If $l_{i+2} = x_4$, it is easy to see that $k \in \{x_1, x_3\}$. If $k = x_1$, noticing that $x_3 \notin V(P)$, then $l_{i+1} \neq x_3$, and thus $x_1x_3 \in E(G)$, a contradiction; if $k = x_3$, then we will have that $|\Gamma_G(x_3) \cap \Gamma_G(x_4)| = 2$, which is impossible in G. (2.) If $l_{i+2} = y_3$, we claim that $k \neq x_3$. Otherwise, we have $y_3x_4 \in E(G)$ and $d(y_3) = 4$, which contradicts the definition of maximal 1-co-belt. Then only $k = x_1$ holds, this implies $|\Gamma_G(x_1) \cap \Gamma_G(y_3)| = 2$, $x_1y_3 \in E(G)$ and $d(x_1) = d(y_3) = 4$, which is impossible in G. Therefore, $d \neq x_2$. By the symmetry of x_2 and x_3, we have that $d \neq x_3$. Therefore, $d = y_2$ holds. Hence $l_{i+2} \in \{x_1, x_4\}$ and $k \in \{y_1, y_3\}$. (1.) If $l_{i+2} = x_1$: We claim that $k \neq y_1$, otherwise, we have $x_1y_1 \in E(G), d(y_1) = 4$, which contradicts the definition of the maximal 1-co-belt. So $k = y_3$ holds. Then we have that $|\Gamma_G(x_1) \cap \Gamma_G(y_3)| = 2$ and $x_1y_3 \in E(G), d(x_1) = d(y_3) = 4$ holds, which is impossible in G. (2.) If $l_{i+2} = x_4$, by the symmetry of x_1 and x_4, we may apply similar arguments to deduce that the assumption is not true.

From the above arguments, we conclude that l_{i+2} is not the end-vertex of P.

We may apply arguments similar to that used in (2) of Theorem 6.2.5 to show that the corollary is true.

(3.) If conclusion (iii) of Lemma 6.2.1 holds, then we have that $B \cap S' = D \cap B = \{d\} \subset \{x_2, x_3, y_2\}$.

We may apply arguments analogous to that used in (2) to show that l_{i+2} is not an end-vertex of P. We may also apply arguments as used in (3) of Theorem 6.2.5 to get that the corollary is true.

(4.) If conclusion (iv) of Lemma 6.2.1 holds, then the corollary is true. This completes the proof. □

6.3 The Number of Removable Edges in a 4-Connected Graph

Being well prepared by the results in the previous section, we are ready to show the main results of this chapter.

Let M be a 5-wheel such that $V(M) = \{a, x, y, z, v\}$ and a is its center. Let T_1, T_2, T_3, T_4 be four trees such that for each $i \in \{1, 2, 3, 4\}$, T_i has k vertices of degree one and $|T_i| - k$ vertices of degree four. Let the vertices of degree four be ${u_i}^{(1)}, {u_i}^{(2)}, \cdots, {u_i}^{(|T_i|-k)}$, and let the vertices of degree one be ${x_i}^{(1)}, {x_i}^{(2)}, \cdots, {x_i}^{(k)}$. Let $M_1, M_2, \cdots, M_k$ be k copies of M and $a^{(j)}, x^{(j)}, y^{(j)}, z^{(j)}, v^{(j)}$ the vertices of M_j corresponding to the vertices a, x, y, z, v of M, respectively, where $j = 1, 2, \cdots, k$. For each $j \in \{1, \cdots, k\}$, identify ${x_1}^{(j)}, {x_2}^{(j)}, {x_3}^{(j)}, {x_4}^{(j)}$ with $x^{(j)}, y^{(j)}, z^{(j)}, v^{(j)}$ such that each of ${x_1}^{(j)}, {x_2}^{(j)}, {x_3}^{(j)}$, ${x_4}^{(j)}$ identifies with one and only one of $x^{(j)}, y^{(j)}, z^{(j)}, v^{(j)}$. Denote the resulting graph by G. It is easy to see that G is 4-connected. Next we show that for each 4-cycle $C = x^{(j)}y^{(j)}z^{(j)}v^{(j)}x^{(j)}$ of G, we have $E(C) \subset E_R(G)$, and the other edges in G are unremovable, where $j = 1, 2, \cdots, k$. For $y^{(j)}{u_i}^{(l)} \in E(G)$, let $S = \{x^{(j)}, v^{(j)}, z^{(j)}\}, A = \{a^{(j)}, y^{(j)}\}, B = G - y^{(j)}{u_i}^{(l)} - S - A$, then $(y^{(j)}{u_i}^{(l)}, S; A, B)$ is a separating group of G, and hence $y^{(j)}{u_i}^{(l)} \in E_N(G)$. Symmetrically, we can show that $x^{(j)}{u_i}^{(l)}, z^{(j)}{u_i}^{(l)}, v^{(j)}{u_i}^{(l)} \in E_N(G)$, where $j = 1, 2, \cdots, k; i = 1, 2, 3, 4; l = 1, 2, \cdots, |T|-k$. For each edge $a^{(j)}x^{(j)}$, it is easy to see that $(a^{(j)}x^{(j)}, T)$ is a separating pair of G such that $T = \{y^{(j)}, v^{(j)}, {u_i}^{(j)}\}$ and ${u_i}^{(l)}z^{(j)} \in E(G)$. By symmetry, we conclude $a^{(j)}y^{(j)}, a^{(j)}z^{(j)}, a^{(j)}v^{(j)} \in E_N(G)$. Using Corollary 2.1.3 it is easy to see that for each 4-cycle $C = x^{(j)}y^{(j)}z^{(j)}v^{(j)}x^{(j)}$, we have $E(C) \subset E_R(G)$. For each edge e of T_i, for example, $e = {u_1}^{(l)}{u_1}^{(l+1)}$, it is easy to see that (e, S) is a separating pair of G such that $S = \{{u_2}^{(l)}, {u_3}^{(l)}, {u_4}^{(l)}\}$. Therefore, for each edge e of T_i, where $i = 1, 2, 3, 4$, we have that $e \in E_N(G)$, and so $e_R(G) = 4k, |T_i| = (3k - 2)/2, (i = 1, 2, 3, 4), |G| = 7k - 4, e_R(G) = (4|G| + 16)/7$. We denote the set of all the above constructed graphs by $\Im$.

Theorem 6.3.1. *Let G be a 4-connected graph of order at least 5. If G is neither C_5^2 nor C_6^2, then $e_R(G) \geq (4|G| + 16)/7$ and the equality holds if and* *only if* $G \in \Im$.

Proof. Let $|G| = n, |E(G)| = m$. We prove the theorem by induction on $(n+m)$. Since G is not C_5^2, we have $n \geq 6$. If $n = 6$, since G is not C_6^2, we obtain $m \geq 13, (n+m) \geq 19$. It is easy to see that $e_R(G) \geq 9 > (4n+16)/7$. For $n = 7$, the relation $e_R(G) \geq 9 > (4n+16)/7$ is also easily seen. Therefore, next we may assume that $n \geq 8$. We distinguish the following cases to complete the proof.

Case 1. G does not have property $(\star)$, i.e., there exists an edge $e = xy \in E_R(G)$ such that $d(x) \geq 5$ and $d(y) \geq 5$ in G.

Then consider $G \ominus e = G - xy$. It is easy to see that removable edges in $G - xy$ are also removable edges in G, and hence $e_R(G) \geq e_R(G \ominus e) + 1$. Then, $|G| = |G \ominus e|, |E(G \ominus e)| = m - 1$, and therefore $|G \ominus e| + |E(G \ominus e)| < n + m$. If $G \ominus e$ is either C_5^2 or C_6^2, then $e_R(G) \geq 9 > (4n+16)/7$. If $G \ominus e$ is neither C_5^2 nor C_6^2, by the induction hypothesis we know that $e_R(G) \geq e_R(G \ominus e) + 1 \geq (4n+16)/7 + 1 > (4n+16)/7$ is true.

Next we may suppose that G has property $(\star)$.

Case 2. G contains a 2-bi-fan as subgraph.

By Theorem 6.2.1 we know that there exists an edge $e \in E(G)$ such that $e_R(G) \geq e_R(G \ominus e) + 1$. Here, $|G \ominus e| = n - 1$, $|E(G \ominus e)| = m - 3$. Then, $|G \ominus e| + |E(G \ominus e)| < n + m$. If $G \ominus e$ equals C_5^2 or C_6^2, then $e_R(G) \geq 10 > (4n+16)/7$. If $G \ominus e$ is neither C_5^2 nor C_6^2, by the induction hypothesis we know that $e_R(G) \geq e_R(G \ominus e) + 1 \geq [4(n-1)+16]/7 + 1 > (4n+16)/7$.

Case 3. G contains an l-belt as its subgraph where $l \geq 3$.

Then by Theorem 6.2.2 there exists an edge $e \in E(G)$ such that $e_R(G) \geq e_R(G \ominus e) + 2$. If $G \ominus e$ is either C_5^2 or C_6^2, then $e_R(G) \geq 12 > (4n+16)/7$. If $G \ominus e$ is neither C_5^2 nor C_6^2, by the induction hypothesis we know that $e_R(G) \geq e_R(G \ominus e) + 2 \geq [4(n-2)+16]/7 + 2 > (4n+16)/7$.

Case 4. For any edge $e \in E_R(G)$, when $|G \ominus e| = n$, we have that $e_R(G) < e_R(G \ominus e)$; when $|G \ominus e| = n-1$, it follows that $e_R(G) < e_R(G \ominus e)+1$; when $|G \ominus e| = n-2$, we deduce that $e_R(G) < e_R(G \ominus e)+2$. Then we distinguish the following subcases.

Subcase 4.1. $[E_N(G)]$ is a forest.

Then $e_N(G) = n - t$ such that t is the number of components in $[E_N(G)]$. Therefore, $e_R(G) \geq 2n - n + t = n + t > (4n + 16)/7$.

Subcase 4.2. $[E_N(G)]$ contains a cycle.

By Theorem 6.2.3 and by the above arguments in Cases 2 and 3 we get that G contains some structures in Υ as its subgraphs. Let G contain k_1 maximal 1-belts, k_2 maximal 1-bi-fans, k_3 maximal 1-co-belts, k_4 W-frameworks, k_5 W'-frameworks, k_6 maximal 2-belts, k_7 maximal 2-co-belts, and h helms. Let E_1 be the set of inner edges of the above-mentioned subgraphs. Then,

$$|E_1| = 2k_1 + k_2 + 3k_3 + 2k_4 + 3k_5 + 4k_6 + 5k_7 + 4h \qquad \langle 1 \rangle$$

Let $E_0 = E_N(G) - E_1$, then we have the following results.

(1.) $[E_0]$ is a forest. This follows from Theorem 6.2.3, Lemma 6.1.1, and the definitions of the above-mentioned subgraphs.

(2.) Let $r = \sum_{x \in G}(d(x) - 4) = \sum_{x \in G} d(x) - 4n$, then $e(G) = 2n + r/2$. Let $n_1 = n - h - |[E_0]|$, then $n_1 \geq 0$, and $n_1 = 0$ if and only if $V(G) = V([E_0]) \bigcup \{a_1, a_2, \cdots, a_h\}$ such that a_i is the center of a helm, where $i = 1, 2, \cdots, h$.

(3.) $e_R(G) = e(G) - e_N(G), e_N(G) = |E_0| + |E_1| = |[E_0]| - t + |E_1| = n - n_1 - h - t + |E_1|$, where t is the number of components in $[E_0]$.

By noticing the number of removable edges in the above-mentioned subgraphs, we have the following result

$e_R(G) = e(G) - e_N(G) = 2n + r/2 - n + h + n_1 + t - |E_1|$

$\geq 3k_1 + 4k_2 + 4k_3 + 5k_4 + 5k_5 + 5k_6 + 6k_7 + 4h.$ ⟨2⟩

From the formulas ⟨1⟩ and ⟨2⟩, we have the following result

$n + r/2 - 7h + n_1 + t - 5k_1 - 5k_2 - 7k_3 - 7k_4 - 8k_5 - 9k_6 - 11k_7 \geq 0.$

Then,

$6n + 3r - 42h + 6n_1 + 6t - 30k_1 - 30k_2 - 42k_3 - 42k_4 - 48k_5 - 54k_6 - 66k_7 \geq 0,$

and so

$e_R(G) = n + r/2 + n_1 + t + h - |E_1| = 4n/7 + (6n + 7r + 14n_1 + 14t - 42h - 28k_1 - 14k_2 - 42k_3 - 28k_4 - 42k_5 - 56k_6 - 70k_7)/14$

$\geq 4n/7 + (6n + 3r + 6n_1 + 6t - 42h - 30k_1 - 30k_2 - 42k_3 - 42k_4 - 48k_5 - 54k_6 - 66k_7)/14 + (4r + 8n_1 + 8t + 2k_1 + 16k_2 + 14k_4 + 6k_5 - 2k_6 - 4k_7)/14$

$\geq 4n/7 + (4r + 8n_1 + 8t + 2k_1 + 16k_2 + 14k_4 + 6k_5 - 2k_6 - 4k_7)/14.$ ⟨3⟩

Therefore, $e_R(G) \geq (4n + 16)/7$ holds only if the following formula holds

$\Delta = 2r + 4n_1 + 4t + k_1 + 8k_2 + 7k_4 + 3k_5 - k_6 - 2k_7 \geq 16.$ ⟨4⟩

Let L'_1 be a maximal 1-co-belt. It is easy to see that $x_2 \in G - \{a_1, a_2, \cdots, a_h\} - V([E_0])$, and so L'_1 will contribute 1 to n_1. Since G contains k_3 maximal 1-belts, they contribute k_3 to n_1. Analogously, for each maximal 2-belt, it contribute 2 to n_1, and so k_6 maximal 2-belts contribute $2k_6$ to n_1. For W'-frameworks, maximal 2-co-belts and W-frameworks, we analyze them analogously. Then we get the following formula

$$n_1 \geq k_3 + k_4 + k_5 + 2k_6 + 3k_7. \quad \langle 5 \rangle$$

From the formulas ⟨5⟩ and ⟨4⟩, wobtain

$$\Delta \geq 2r + 4t + k_1 + 8k_2 + 4k_3 + 11k_4 + 7k_5 + 7k_6 + 10k_7. \quad \langle 6 \rangle$$

We now discuss the following cases.

(4.) $h = 0, k = k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 \le 2$.

First, we claim that $[E_N(G)]$ contains at most two cycles. Otherwise, suppose that there are at least three cycles in $[E_N(G)]$. Then we consider a cycle C_1. By Theorem 6.2.4 and the assumption, we conclude that G contains some structure $H_1 \in \Upsilon$ as its subgraph such that H_1 has an inner edge e_1 on C_1. We consider another cycle C_2 in $[E_N(G)] - C_1$. Analogously, we have that G contains some structure $H_2 \in \Upsilon$ as its subgraph such that H_2 has an inner edge e_2 on C_2. Last, we consider a cycle C_3 in $[E_N(G)] - C_1 - C_2$. Then, G contains some structure $H_3 \in \Upsilon$ as its subgraph such that H_3 has an inner edge e_3 on C_3. Since e_1 is an inner edge of H_1, but not of H_2, we have $H_1 \neq H_2$. Analogously, $H_1 \neq H_3, H_2 \neq H_3$. By Lemma 6.1.1 we know that any two of H_1, H_2 and H_3 do not have common inner edges, and so $k \geq 3$, a contradiction. Therefore, there are at most two cycles in $[E_N(G)]$. So, $e_N(G) \le n+1$, and hence $e_R(G) \geq 2n - n - 1 > (4n+16)/7$.

(5.) $h = 0, k = k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 \geq 3$.

(5.1.) $k_1 + k_3 = 0$, and so $k_2 + k_4 + k_5 + k_6 + k_7 \geq 3$. Noticing that $t \geq 1$, from formula $\langle 6 \rangle$ we obtain

$\Delta \geq 2r+4+7(k_2+k_4+k_5+k_6+k_7)+k_2+4k_4+3k_7 \geq 4+7(k_2+k_4+k_5+k_6+k_7) \geq 25$,

here the inequality $\langle 4 \rangle$ rigidly holds.

(5.2.) $k_1 + k_3 \geq 1$. We may assume that G contains a maximal 1-belt L_1 such that $V(L_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. By Theorem 6.2.5 we know that if $x_3, y_1 \in [E_0]$, then $n_1 \geq 2, t \geq 2$. By the formulas $\langle 4 \rangle$ and $\langle 5 \rangle$ we see that

$\Delta \geq 2r + 3n_1 + 4t + (k_1 + k_3) + 8k_2 + 8k_4 + 4k_5 + k_6 + k_7 \geq 3n_1 + 4t + (k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7) \geq 6 + 8 + 3 = 17$.

If $x_3 \in [E_0], y_1 \notin [E_0]$, then $n_1 \geq 1, t \geq 3$. Similarly, we get $\Delta \geq 18$.

If $x_3, y_1 \in [E_0]$, then $t \geq 4$, and so $\Delta \geq 19$, here the inequality $\langle 4 \rangle$ rigidly holds.

(6.) $h \geq 1$. We take a helm H such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$. By Theorem 6.2.4 we have that any two of the edges $x_1v_1, x_2v_2, x_3v_3, x_4v_4$ are in different components, and so $t \geq 4$. By formula $\langle 6 \rangle$ we know $\Delta \geq 16$, i.e., $e_R(G) \geq (4n + 16)/7$, and the equality holds only if $k_i = 0$, where $i = 1, 2, \cdots, 7, r = 0, t = 4, n_1 = 0$, i.e., $[E_0]$ has only four components T_1, T_2, T_3, T_4, and $V(G) = V([E_0]) \cup \{a_1, a_2, \cdots, a_h\}$. Then from $r = 0$ we know that G is a 4-connected and 4-regular graph. From $e_R(G) = 4h, e_N(G) = 10h - 8$, we conclude that $n = 7h - 4$. Moreover, all edges except for ${x_1}^{(p)}{x_2}^{(p)}, {x_2}^{(p)}{x_3}^{(p)}, {x_3}^{(p)}{x_4}^{(p)}, {x_4}^{(p)}{x_1}^{(p)}$ of each helm H_p in G are unremovable, whereas different edges of ${x_i}^{(p)}{v_i}^{(p)}$ of H_p are in different components T_i, and every vertex ${v_i}^{(p)}$ is of degree 1 in T_i. Based on the above arguments, we conclude that T_i has h vertices with degree 1 and $|T_i| - h$ vertices with degree 4. Therefore, $G \in \Im$. This complete the proof. $\Box$

Chapter 7

Removable Edges in a Spanning Tree or outside a Cycle in a 4-Connected Graph

In this chapter we study the distribution of removable edges in a spanning tree or outside a cycle in a 4-connected graph. We give examples to show that our results are best possible in some sense.

7.1 Removable Edges on a Spanning Tree

Before we give the main result, we first show the following lemma.

Lemma 7.1.1. *Let G be a 4-connected graph without any subgraph belonging to $\Re$, and let $[E_N(G)]$ be a tree. Then $|[E_N(G)]| \leq |G| - 3$.*

Proof. By contradiction. Assume $|[E_N(G)]| \geq |G| - 2$. Let x be a vertex of degree 1 in the tree $[E_N(G)]$. Since $d_G(x) \geq 4$ and $|[E_N(G)]| \geq |G| - 2$, there is a vertex $y \in [E_N(G)]$ such that $xy \in E_R(G)$. Let P be a path joining x and y in $[E_N(G)]$, then $P + xy$ is a cycle of G that contains precisely one removable edge xy. We consider the cycle $C = y_1y_2 \dots y_ky_1$ such that $y_1y_k \in E_R(G), E(C) - \{y_1y_k\} \subset E_N(G)$, and the length of C is as small as possible in G.

Let $D = \{y_1\}$. Consider the path $P = y_1y_2 \dots y_k$ in $[E_N(G)]$. Consider a separating group $(y_1y_2, S_1; A_1, B_1)$ such that $y_1 \in B_1, y_2 \in A_1$. Obviously, $D \cap B_1 \neq \emptyset$. We take $i \in \{1, 2, \ldots, k-1\}$ and a separating group $(y_iy_{i+1}, S; A, B)$ such that $y_i \in B, y_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is as small as possible. We claim $i+1 \leq k-1$. Otherwise, $i+1 = k$, then $y_k \in A$, since $y_1y_k \in E(G)$ and $y_1 \in A \cup S$, which contradicts $D \cap B \neq \emptyset$. So, $i+1 \leq k-1$. Then we take the separating group $(y_{i+1}y_{i+2}, S'; A', B')$ such that $y_{i+1} \in B', y_{i+2} \in A'$ and $|A'|$ is small as possible. By Lemma 6.2.1 we have that one of conclusions (i), (ii), (iii) or (iv) of Lemma 5.1.1 holds.

(1.) Conclusion (i) holds. Since $y_1 \in B$, we have $y_k \in B \cup S$. So y_{i+2} is not an end-vertex of P, and so y_{i+2} associates with at least two unremovable edges in G, which contradicts conclusion (i) of Lemma 6.2.1.

(2.) Conclusion (ii) holds. Then $d = y_1$. We let $C' = A', T' = A \cap S' \cup \{y_{i+1}\}$ and $D' = G - cd - T' - C'$. Then $(cd, T'; C', D')$ is a separating group of G, and so $cd \in E_N(G)$. Since $y_1y_k \in E_R(G)$, we have $c \neq y_k$. Hence $y_k \in B' \cap (B \cup S)$. Let $A \cap S' = \{u, v\}$. Since y_{i+2} is not an end-vertex of P, we have $\{cy_{i+2}, uy_{i+2}, vy_{i+2}\} \cap E_N(G) \neq \emptyset$. From Corollary 3.1.1 we know that y_{i+2} is a vertex of some subgraph belonging to $\Re$, which contradicts the assumptions. Hence, conclusion (ii) does not occur.

(3.) Conclusion (iii) holds. First we let $C' = A', T' = (S' - \{d\}) \cup \{y_{i+1}\}$ and $D' = G - cd - T' - C'$. Then $(cd, T'; C', D')$ is a separating group of G. So $cd \in E_N(G)$, and hence $c \neq y_k$. Obviously, y_{i+2} is not an end-vertex of P. By an analogous argument as used in (2.), we can deduce that conclusion (iii) does not occur.

(4.) From the assumption of the theorem, we know that conclusion (iv) does not occur.

This complete the proof of Lemma 7.1.1. $\square$

Theorem 7.1.1. *Let G be a 4-connected graph which does not contain any subgraph belonging to $\Re$. Then any spanning tree T of G contains at least one removable edge.*

Proof. First, we claim that $[E_N(G)]$ does not contain any cycle. Otherwise, if $[E_N(G)]$ contains a cycle. By Theorem 6.2.3 and its proof we know that G must contain some a subgraph belonging to $\Re$, which contradicts the assumption of the theorem.

If $[E_N(G)]$ is a tree, then by Lemma 7.1.1 we have $|[E_N(G)]| \leq |G| - 3$. Since $|E(T)| = |G| - 1$, we have $|E(T) \cap E_R(G)| \geq 2$.

If $[E_N(G)]$ is a forest with at least two components, then clearly the theorem holds. $\square$

Here we give an example to show that the low bound of the theorem is sharp.

Example 7.1.1. Let H be a helm as in Definition 2.1, such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$.

Let $L = H - \{v_1, v_2, v_3, v_4\}$, and L' be a copy of L such that $V(L') = \{a', x_1', x_2', x_3', x_4'\}$. We construct a graph G as follows:

Let $V(G) = V(L) \cup V(L')$ and $E(G) = E(L) \cup E(L') \cup \{x_1x_3, x_2'x_4', x_1x_1', x_2x_2', x_3x_3', x_4x_4'\}$. Obviously, G is a 4-connected graph which does not contain any subgraph belonging to $\Re$. It is easy to see that $(ax_2, \{x_1, x_3, x_4'\})$ is a separating pair of G, and so $ax_2 \in E_N(G)$. By symmetry, $ax_4, a'x_1', a'x_3' \in E_N(G)$. Similarly, $(x_1x_1', \{x_2, x_3, x_4\})$ is a separating pair of G, and hence $x_1x_1' \in E_N(G)$. By symmetry, we have $x_2x_2', x_3x_3', x_4x_4' \in E_N(G)$.

Let T be a spanning tree of G such that $E(T) = \{x_1x_1', x_2x_2', x_3x_3', x_4x_4', a'x_2', a'x_3', ax_1', ax_2, ax_4\}$. Then it is easily checked that there is only one removable edge $a'x_2'$ in T.

7.2 Removable Edges outside Cycles

In this section we study the distribution of removable edges outside a cycle in a 4-connected graph. First, we give the following definition.

Definition 7.1.1. Let C be a cycle of a 4-connected graph G and H a subgraph of G belonging to $\Re$. If C contains an inner vertex of H, then we say that C *passes through* H.

We prove two results on the existence of removable edges outside a cycle.

Theorem 7.2.1. *Let G be a 4-connected graph and C be a cycle of G. If C does not pass through any l-belt or l-co-belt, then there are at least two removable edges outside C.*

Proof. By contradiction. Assume that there is at most one removable edge outside C. Let $F = (E(G) - E(C)) \cap E_R(G)$. Then $|F| \leq 1$. We denote $E(G) - E(C) - F$ by E_0. If C does not pass through any subgraph belonging to $\Re$, we take the separating group $(uw, S'; A', B')$ such that $u \in A', w \in B'$ and $uw \in E_0$. Since $|F| \leq 1$, we know that either $(E(A') \cup [A', S']) \cap F = \emptyset$ or $(E(B') \cup [S', B']) \cap F = \emptyset$ hold. Without loss of generality, we may assume $(E(A') \cup [A', S']) \cap F = \emptyset$. If C passes through a subgraph H of G that belongs to $\Re$, noticing that H is neither an l-belt nor an l-co-belt, then from the definition of $\Re$ we have the following discussion: If H is a maximal l-bi-fan ($l \geq 1$), then $l = 1$. If $C \subset E(H)$, then according to the assumption we have $e_R(G) \leq 5$. Obviously, $|G| \geq 7$. By Theorem 6.3.1 we have $e_R(G) \geq (4|G| + 16)/7 > 5$, a contradiction. So we have $F \cap E(H) \neq \emptyset$. By letting $S' = \{a, b, x_4\}, e = x_2x_1, B' = \{x_2, x_3\}$ and $A' = G - e - S' - B'$, we get that $(e, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of the maximal l-bi-fan, and $F \cap (E(A') \cup E([A', S'])) = \emptyset$. If H is a helm, we use a similar argument to get $F \cap E(H) \neq \emptyset$. By letting $e = x_1v_1, S' = \{v_2, v_3, v_4\}, B' = \{a, x_1, x_2, x_3, x_4\}$ and $A' = G - e - S' - B'$, we get that $(e, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of the helm, and $F \cap (E(A') \cup E([A', S'])) = \emptyset$. If H is a W-framework, according to the assumption we must have $F = y_2y_3$. In this case, by letting $e = x_2x_1, S' = \{x_3, y_4, y_2\}, B' = \{x_2, y_3\}$ and $A' = G - e - S' - B'$, we get that $(e, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of the W-framework, and $F \cap (E(A') \cup E([A', S'])) = \emptyset$. If H is a W'-framework, according to the assumption we have $E(C) \subset E(H)$ and $F \subset E(H)$. Then, $e_R(G) = 5$. However, by Theorem 6.3.1 we have $e_R(G) \geq (4|G| + 16)/7 > 5$, a contradiction.

Since A' is an E_0-edge-vertex cut fragment, A' must contain an E_0-edge-vertex cut end-fragment as its subgraph, say A. Then we have $(E(A) \cup [A, S]) \cap F = \emptyset$, and we take the corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$.

We distinguish the following cases to complete the proof.

Case 1. $|A| = 2$.

Then, either A is a 1-edge-vertex cut atom or a 2-edge-vertex-cut atom. Let $A = \{x, z\}$ and $S = \{a, b, c\}$. If A is a 1-edge-vertex-cut atom, and $\{xz, xa, xb\} \cap E_N(G) \neq \emptyset$, from Corollary 3.1.1 we have that x is an inner vertex of some subgraph belonging to $\Re$, a contradiction. So, we have $xz, xa, xb \in E_R(G)$. Since C is a cycle, then $F \cap E(A) \neq \emptyset$, a contradiction. Suppose now that A is a 2-edge-vertex-cut atom, it is easily checked that $F \cap (E(A) \cup [A, S]) \neq \emptyset$. The theorem holds.

Case 2. $|A| \geq 3$.

Since C is a cycle of G, and $(E(A) \cup [A, S]) \cap F = \emptyset$, there exists $xz \in E_0 \cap E(A \cup [A, S])$. Obviously, $z \notin S$, otherwise we would have $|A| = 2$. We take the separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1, z \in B_1$. Then we have $x \in A \cap A_1$ and $z \in A \cap B_1$. Hence one of the three conclusions of Lemma 5.1.1 holds.

(1.) Since C is a cycle and $(E(A) \cup [A, S]) \cap F = \emptyset$, conclusion (i) does not occur.

(2.) Suppose that conclusion (ii) holds. Since B' is a 1-edge-vertex-cut atom, by noticing $F \cap (E(A) \cup E([A, S])) = \emptyset$, we can use a similar argument as used in Case 1 to show that the conclusion of the theorem holds.

(3.) Finally, suppose that conclusion (iii) holds. Let $B' \cap S = \{x_1, x_2\}$ and $A \cap B' = \{x_3\}$. By combining the conclusion of Lemma 5.1.2, we have $zx_1, zx_2 \in E_R(G)$. Since $|B'| \geq 3$, from Theorem 2.1.2 we have $y'x_3 \in E_R(G)$. Note that C is a cycle of G, and $F \cap (E(A) \cup [A, S]) = \emptyset$, which is impossible to hold. So, the theorem holds.

This complete the proof. $\square$

We give an example to show that the result of Theorem 7.2.1 is best possible.

Example 7.2.2. Let H be a helm as in Definition 1.2.1, $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$, $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$.

Let $L = H - \{v_1, v_2, v_3, v_4\}$, and L' be a copy of L such that $V(L') = \{a', x'_1, x'_2, x'_3, x'_4\}$. We construct a graph G as follows:

Let $V(G) = V(L) \cup V(L')$ and $E(G) = E(L) \cup E(L') \cup \{x_1x'_1, x_2x'_2, x_3x'_3, x_4x'_4\}$. Obviously, G is a 4-connected graph. It is easy to see that $(ax_2, \{x_1, x_3, x'_4\})$ is a separating pair of G, and so $ax_2 \in E_N(G)$. By symmetry, $ax_4, ax_1, ax_3, a'x'_1, a'x'_2, a'x'_3, a'x'_4 \in E_N(G)$. It is easy to see that $(x_1x'_1, \{x_2, x_3, x_4\})$ is a separating pair of G, and hence $x_1x'_1 \in E_N(G)$. By symmetry, we have $x_2x'_2, x_3x'_3, x_4x'_4 \in E_N(G)$.

Let C be the cycle $x_1x_4x_3x_2x'_2x'_3x'_4x'_1x_1$. Clearly, C does not pass through any l-belt or l-co-belt. It is easy to see that C is a cycle outside which there are exactly two removable edges $x_1x_2, x'_1x'_2$.

Theorem 7.2.2. *Let G be a 4-connected graph and C a cycle of G. If C passes through only one (maximal) l-belt or l-co-belt, then there is at least one removable edge outside C.*

Proof. By contradiction. Assume that there is no removable edge outside C. Let $E_0 = E(G) - E(C)$. Suppose that C passes through a subgraph H in $\Re$. If H is one of the following structures: helm, W'-framework, W-framework or l-bi-fan, then we can apply similar arguments as used in the proof of Cases 1 and 2 of Theorem 7.2.1 to show that the conclusion is true. So, we may suppose that H is either a (maximal) l-belt or l-co-belt. If H is a maximal l-belt as in Definition 1.2.3, from the assumption it is easy to see that $E_0 \subset E(C)$, and $x_2x_1 \in E_0$. By letting $S = \{y_{l+2}, x_{l+2}, y_1\}, e = x_2x_1, B = \{x_2, \cdots, x_{l+1}, y_2, \cdots, y_{l+1}\}$, $A = G - e - S - B$, we get that $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H, if H is a maximal l-co-belt as in Definition 1.2.4, similarly, we have $x_1x_2 \in E_0$. By letting $S = \{y_{l+2}, x_{l+3}, y_1\}, e = x_2x_1, B = \{x_2, \cdots, x_{l+2}, y_2, \cdots, y_{l+1}\}$ and $A = G - e - S - B$, we get that $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of H. We can apply similar arguments as used in Cases 1 and 2 of Theorem 7.2.1 to complete the proof of the theorem. $\Box$

We complete this chapter by presenting two examples to show that if a cycle C in a 4-connected graph G contains two l-belts or l-co-belts, then there could be no removable edge outside C of G. So in this sense the condition of the Theorem 7.2.2 is best possible.

Example 7.2.3. Let H be an l-belt as in Definition 7.2.3, H' be a copy of H such that $V(H') = \{x'_1, x'_2, \cdots, x'_{l+2}, y'_1, y'_2, \cdots, y'_{l+2}\}$ and $E(H') = E_1(H') \cup E_2(H')$ where $E_1(H') = \{x'_1x'_2, x'_2x'_3, \cdots, x'_{l+1}x'_{l+2}, y'_1y'_2, y'_2y'_3, \cdots, y'_{l+1}y'_{l+2}\}$ and $E_2(H') = \{y'_1x'_2,\ x'_2y'_2,\ y'_2x'_3, \cdots, y'_lx'_{l+1}, x'_{l+1}y'_{l+1}, y'_{l+1}x'_{l+2}\}$. Identify vertex x_1 with y'_1, vertex y_1 with y'_{l+2}, vertex y_{l+2} with x'_{l+2}, vertex x_{l+2} with x'_1. Then join vertex x_{l+2} with y'_1, vertex y_1 with x'_{l+2}. Denote the resulting graph by G. It is easily checked that G is a 4-connected graph and $(x_2y'_1, \{y_1, x_3, y_3\})$ is a separating group of G, hence $x_2y'_1 \in E_N(G)$. Similarly, we can show $\{y_1y'_{l+1}, y_{l+1}x'_{l+2}, x'_2x_{l+2}\} \subset E_N(G)$. Let C be the cycle $y_1x_2y_2x_3 \ldots x_{l+1}y_{l+1}x_{l+2}$

$y_1'x_2'y_2'\ldots x_{l+1}'y_{l+1}'x_{l+2}'y_1$. Then it is easy to see that there is no removable edge outside C.

Example 7.2.4. Let H be an l-co-belt as in Definition 1.2.4, H' be a copy of H such that $V(H') = \{x_1', x_2', \cdots, x_{l+2}', x_{l+3}', y_1', y_2', \cdots, y_{l+2}'\}$ and $E(H') = E_1(H') \cup E_2(H')$, where $E_1(H') = \{x_1'x_2', x_2'x_3', \cdots, x_{l+1}'x_{l+2}', x_{l+2}'x_{l+3}', y_1'y_2', y_2'y_3', \cdots, y_{l+1}'y_{l+2}'\}$ and $E_2(H') = \{y_1'x_2', x_2'y_2', y_2'x_3', \cdots, x_{l+1}'y_{l+1}', y_{l+1}'x_{l+2}', x_{l+2}'y_{l+2}'\}$. First, delete vertices x_1, x_1', x_{l+3}, x_{l+3}' from H and H', respectively. Then join vertex x_{l+2} with y_{l+2}', vertex y_1 with y_1', vertex x_{l+2}' with y_{l+2}, vertex x_2 with x_2', vertex y_1 with y_{l+2}', vertex y_1' with y_{l+2}, respectively. Denote the resulting graph by G. It is easily checked that G is a 4-connected graph, and $(y_1y_1', \{y_{l+2}, y_{l+2}', x_2'\})$ is a separating group of G, and so $y_1y_1' \in E_N(G)$. Similarly, $(x_{l+2}y_{l+2}', \{y_l, x_{l+1}, y_{l+2}\})$, $(x_2x_2', \{y_1, x_3, y_3\})$ and $(y_{l+2}x_{l+2}', \{x_{l+1}', y', y_{l+2}'\})$ are separating groups of G, and so $\{x_{l+2}y_{l+2}', x_2x_2', y_{l+2}x_{l+2}'\} \subset E_N(G)$. Let C be the cycle $y_1x_2y_2x_3\ldots x_{l+1}y_{l+1}x_{l+2}y_{l+2}y_1'x_2'y_2'\ldots x_{l+1}'y_{l+1}'x_{l+2}'y_{l+2}'y_1$. Then it is easy to see that there is no removable edge outside C.

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